MA 523: Partial Differential Equations
January 2015, Qualifying Examination (Yip)

Your PUID: ____________________________

This examination contains five questions, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Let Ω be a smooth bounded domain in $\mathbf{R}^2$. Let also $V(x, y) = (A(x, y), B(x, y))$ be a smooth vector field defined on Ω such that $V \cdot \hat{n} > 0$ on $\partial\Omega$ (where $\hat{n}$ is the outward normal to $\partial\Omega$).

   Suppose u is a smooth solution of the following equation on the whole Ω:

   $$A(x, y)u_x + B(x, y)u_y = -u.$$

   Show that u vanishes identically.

   (Hint: investigate the behavior of u at its interior and boundary maxima and minima.)

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2. Solve the following PDE:

$$\begin{aligned} u_{yy} &= u_{xx} + u, \\ u(x,0) = e^x, &\qquad u_y(x,0) = 0. \end{aligned}$$

(Hint: use power series expansion in the y-variable with x-dependent coefficients or use separation of variables.)

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3. Solve the following wave equation on the whole real line:

$$\begin{aligned} u_{tt} - u_{xx} &= x^2, \quad \text{for } 0 < t,\ -\infty < x < \infty, \\ u = x, \qquad & u_t = 0, \quad \text{for } t = 0 \end{aligned}$$

It is not sufficient to just write down a general formula. Compute all the necessary integrals if there are any.

(Hint: it might be easier to first find a *special time independent* solution.)

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4. Consider the following heat equation on the whole real line:

$$
\begin{aligned}
u_t &= u_{xx}, \quad -\infty < x < \infty, \;\; t > 0, \\
u(x,0) &= f(x)
\end{aligned}
$$

Prove the following estimates:

(a) $\|u\|_{L^\infty} \leq C\,\|f\|_{L^\infty}$,

(b) $\|u\|_{L^\infty} \leq C\dfrac{\|f\|_{L^1}}{\sqrt{t}}$,

(c) $\|u_x\|_{L^\infty} \leq C\dfrac{\|f\|_{L^\infty}}{\sqrt{t}}$,

(d) $\|u_x\|_{L^\infty} \leq C\dfrac{\|f\|_{L^1}}{t}$,

(e) $\|u_x\|_{L^\infty} \leq C\,\|f_x\|_{L^\infty}$,

(f) $\|u_x\|_{L^\infty} \leq C\dfrac{\|f_x\|_{L^1}}{\sqrt{t}}$.

In the above, C is some constant and the spaces L^∞, L^1 are defined with respect to the spatial variable x. You can assume all the functions are nice and smooth and can arbitrarily interchange integration and differentiation.

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5. Let $u(x,t)$ be a positive solution of the following equation:

$$u_t = \mu u_{xx}, \quad \text{for } t > 0$$

where μ is some positive constant. You are given the fact that the function $v(x,t) = -2\mu \frac{u_x}{u}$ solves the following "viscous" Burgers' equation:

$$v_t + v v_x = \mu v_{xx} \quad \text{for } t > 0.$$

Let the initial data for v be given by $v(x,0) = \phi(x) \in C_0(\mathbf{R})$ (i.e. ϕ has compact support).

(a) Show that shocks for v will not form. (This is in contrast with the inviscous Burgers' equation ($\mu = 0$), $v_t + v v_x = 0$.)

(b) Show that for some constant C,

$$\|v(\cdot,t)\|_{L^\infty} \le \frac{C}{\sqrt{t}} \quad \text{(and hence } \lim_{t\to\infty} v(x,t) = 0 \text{ uniformly in } x\text{).}$$

(Hint: first relate the initial data of u to ϕ and then use Green's function representation for $u(x,t)$.)

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