MATH 530 Qualifying Exam
January 2017 (S. Bell)

Each problem is worth 20 points

1. Suppose that \( f(z) \) is analytic on the complex plane minus a single point \( z_0 \). Suppose further that \( f \) has a simple pole at \( z_0 \) and a removable singularity at infinity. Prove that
\[
f(z) = \frac{A}{z - z_0} + B,
\]
where \( A \) and \( B \) are complex constants.

2. Let
\[
f(z) = \frac{\log z}{(z^2 + 4)^2},
\]
where \( \log \) denotes a branch of the complex logarithm with branch cut along the negative imaginary axis that agrees with the real logarithm \( \ln \) on the positive real axis. For a radius \( r > 0 \), let \( C_r \) denote the half circle parametrized by \( z(t) = re^{it} \) for \( 0 \leq t \leq \pi \), and for \( a < b \), let \( L[a,b] \) denote the line segment on the real line parametrized by \( z(t) = t \) for \( a \leq t \leq b \).

a) Assume that \( r > 0 \). Prove that \( \int_{C_r} f(z) \, dz \) goes to zero as \( r \) goes to infinity and as \( r \) goes to zero.

b) Assume that \( 0 < \epsilon < R \). Note that \( \int_{L[\epsilon,R]} f(z) \, dz = \int_{\epsilon}^{R} \frac{\ln t}{(t^2 + 4)^2} \, dt \). Express \( \int_{L[-R,-\epsilon]} f(z) \, dz \) in terms of explicit real integrals.

c) Compute the residue of \( f(z) \) at \( 2i \).

d) Finally, use the residue theorem, take limits, and take the real part to compute
\[
I = \int_{0}^{\infty} \frac{\ln t}{(t^2 + 4)^2} \, dt.
\]

3. Suppose that \( \{a_n\}_{n=1}^\infty \) is a sequence of distinct points in the unit disc with no limit points in the disc. Prove that the radius of convergence of the power series \( \sum_{n=1}^{\infty} a_n z^n \) is equal to one.

4. Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{(z-n)^2} \) converges on the complex plane minus the positive integers to an analytic function with a double pole at each positive integer.

5. Suppose that \( f(z) \) is a continuous complex valued function on a disc such that the integral \( \int_{\gamma} f(z) \, dz \) is equal to zero for every contour \( \gamma \) that is the boundary of a square in the disc. Prove that \( f \) must be analytic.