MA 53000 QUALIFIER, 1/3/2017

Each problem is worth 5 points. Make sure that you justify your answers.

Notes, books, crib sheets, and electronic devices are not allowed.

1. f is a function holomorphic in the half plane $\{\operatorname{Im} z > -3\}$, apart from a simple pole at $z = 2$. What can you say about the radius of convergence of its Taylor series about 0? Same question if the simple pole is, instead, at $z = 4$.

2. Compute the following integral (the path of integration is oriented counter-clockwise):

$$\int_{|z|=2} \frac{e^z}{z - z^2}\, dz.$$

3. Suppose ϕ is holomorphic on some open set $\Omega \subset \mathbb{C}$, apart from isolated singularities. Suppose furthermore that for each $k \in \mathbb{N}$ we can write $\phi = \psi^k$ with a ψ that is also holomorphic on Ω, apart from isolated singularities. Prove that the singularities of ϕ are either removable or essential.

4. For positive numbers a, R let $\Gamma_{a,R} \subset \mathbb{C}$ stand for the path consisting of three segments as follows. It starts at $R - \pi i$, goes to $a - \pi i$, from there to $a + \pi i$ and then to $R + \pi i$. Prove that

$$\lim_{R\to\infty} \int_{\Gamma_{a,R}} \frac{e^{e^\zeta}}{\zeta - z}\, d\zeta = E_a(z)$$

exists and represents a holomorphic function E_a in the half plane $H_a = \{z \in \mathbb{C} \colon \operatorname{Re} z < a\}$. Prove also that if $a < b$ then $E_a = E_b$ in H_a.

5. Find a biholomorphic map between the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and the half disc $\{z \in D : \operatorname{Im} z > 0\}$. (If the map is found as the composition of simpler maps, it suffices to explain what the simpler maps are, there is no need to write down the composition.)

6. Suppose u is a harmonic function in $\mathbb{C}$, and $|u(z)| \le \sqrt{|z|}$ for $z \in \mathbb{C}$. Prove that u is constant.

7. Prove that if P is holomorphic on $\mathbb{C}$ and $\lim_{|z|\to\infty} |P(z)| = \infty$, then P is a polynomial.

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