MA 53000 QUALIFIER, 8/12/2016

Each problem is worth 5 points. Make sure that you justify your answers. The set of holomorphic functions on an open $\Omega \subset \mathbb{C}$ is denoted $\mathcal{O}(\Omega)$. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Notes, books, crib sheets, and electronic devices are not allowed.

1. In what annuli (including degenerate ones), centered at 0, can we expand the function $\dfrac{3}{z(z+3)}$ in a Laurent series? Find the expansion in those annuli.

2. Compute the following integral (the path of integration is oriented counterclockwise, and log stands for the branch of logarithm on $\{z : \operatorname{Re} z > 0\}$ for which $\log 1 = 0$):

$$\int_{|z-1|=1/2} \frac{z \log z}{(z-1)^2}\, dz.$$

3. If f is holomorphic on some open set in $\mathbb{C}$, show that the product $\operatorname{Re} f(z) \operatorname{Im} f(z)$ is harmonic there.

4. Suppose $\Omega \subset \mathbb{C}$ is open, $g \in \mathcal{O}(\Omega \setminus \{a\})$ for some $a \in \Omega$, and $\operatorname{Re} g \geq 0$ everywhere. Prove that then the singularity of g at a is removable.

5. Let $[0,1] \subset \mathbb{C}$ stand for the closed interval between 0 and 1, and $\Omega = \big(\mathbb{C} \setminus [0,1]\big) \cup \{\infty\} \subset \hat{\mathbb{C}}$. Find a biholomorphic map of Ω on the unit disc.

6. Suppose $\Omega \subset \mathbb{C}$ is open, and $F_n \in \mathcal{O}(\Omega)$ for $n \in \mathbb{N}$ are uniformly bounded. If $F_n(z)$ converge for all $z \in \Omega$, prove that $\lim_{n\to\infty} F_n \in \mathcal{O}(\Omega)$.

7. Let φ be a holomorphic function on the disc $\{z : |z| < 1\}$ such that $\operatorname{Re} \varphi(z) > 0$ when $|z| < 1$, and $\varphi(0) = 1$. Prove that

$$|\varphi(z)| \leq \frac{1+|z|}{1-|z|} \qquad \text{when } |z| < 1.$$