QUALIFYING EXAMINATION

AUGUST 1994

MATH 530

All answers must be justified and work must be shown.

1. Let f be an analytic function in the open unit disk, $|f(z)| \leq 1, \quad |z| < 1$. Prove that $|f^{(n)}(0)| \leq n!; \quad n = 0, 1, 2, \ldots$.

2. Let f be a non-constant analytic function in a neighborhood N of the real axis $\mathbb{R}$. Assume that
$$\operatorname{Im} f(z) \cdot \operatorname{Im} z \geq 0, \quad z \in N.$$
   a) Show that $f'(z) \neq 0, \quad z \in \mathbb{R}$.
   b) Show that actually $f'(z) > 0, \quad z \in \mathbb{R}$.

3. Evaluate the integral
$$\int_0^\infty \frac{x^{\alpha-1} dx}{x+t},$$
   where $0 < \alpha < 1$ and $t > 0$.

4. Find the one-to-one conformal map of the region $\{z : \operatorname{Re} z > 0, \quad \operatorname{Im} z > 0, |z| > 1\}$ onto the upper half-plane, such that $i \mapsto 0$, $1 \mapsto 1$ and $\infty \mapsto \infty$.

5. How many solutions (counting multiplicity) on the Riemann sphere can have the equation
$$f(z) - z = 0,$$
   where f is a rational function of degree $d \geq 2$.

   (The degree of $f = P/Q$ is defined as $\max\{\deg P, \deg Q\}$, where P and Q are polynomials without common factor.)

6. Describe the set in the complex plane where $\cos z$ is real. Draw the picture.

7. Find the residus of $\cot^2 z$ at all isolated singular points.