QUALIFYING EXAMINATION
JANUARY 2001
MATH 530 - Prof. Eremenko

1. Suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n z^{-n}$$

is a Laurent series at infinity of a rational function f, which has only simple poles. Prove that there exist an integer m and complex numbers c_j, b_j, such that

$$a_n = \sum_{j=1}^{m} c_j b_j^n, \quad n = 1, 2, 3, \ldots.$$

2. Find the principal value of the integral

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^3}\, dx.$$

3. Suppose that u is a harmonic function in the region $\{z : |z| < 1, \operatorname{Im} z > 0\}$, continuous in the closure of this region, and having the property

$$\lim_{y \to 0+} u(x + iy)/y = 0, \quad \text{for all} \quad x \in (-1, 1),$$

Prove that $u \equiv 0$.

4. Recall that a map $f : X \to Y$ is called *surjective* if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
a) Does there exist a surjective holomorphic map of the complex plane onto the unit disc?
b) Does there exist a surjective holomorphic map of the unit disc onto the complex plane?
If your answer is “yes”, you have to give an example, if it is “no”, you have to prove that such map does not exist. In your proof, you can use any fact, proved in Ahlfors’ book, but you have to state this fact precisely.

5. a) What is the radius of convergence of the series in the right hand side of this equation:

$$\frac{\operatorname{Log} z}{z - 1} = \sum_{n=0}^{\infty} a_n (z - 2)^n \quad ?$$

Here Log is the principal branch of the logarithm, $\mathrm{Log}(z) = \mathrm{Log}|z| + i\mathrm{Arg}z$, where $|\mathrm{Arg}z| < \pi$.
b) Will your answer change, if we consider some other branch of the logarithm in the same region $|\mathrm{Arg}z| < \pi$?

6. Does there exist an analytic function f in the unit disc, with the following property

$$f(1/n) = \frac{n}{2+n} \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Give an example or prove that it does not exist.