1. (20 pts) Suppose that \( f \) is a continuous complex valued function on the unit disc \( D_1(0) \) and that \( f \) is analytic on the upper half disc, \( \{ z : \text{Im} \ z > 0 \} \cap D_1(0) \), and analytic on the lower half disc, \( \{ z : \text{Im} \ z < 0 \} \cap D_1(0) \). Use only Morera’s Theorem to prove that \( f \) must actually be analytic on the whole disc.

2. (20 pts) Evaluate the integral \( \int_0^\infty \sin(x^2) \, dx \) by integrating \( e^{iz^2} \) around the contour that starts at the origin and follows the real line out to a point \( R > 0 \), then follows the circular arc \( Re^{it} \) from \( t = 0 \) to \( t = \pi/4 \), and returns to the origin along the line joining \( Re^{i\pi/4} \) to 0. Let \( R \to \infty \). You may use the fact that \( \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}/2 \) without proving it.

3. (20 pts) Prove that the integral
\[
\int_0^1 \frac{t^2}{t-z} \, dt
\]
defines an analytic function \( f(z) \) on \( \mathbb{C} - [0, 1] \). State Liouville’s Theorem and use it to prove that \( f \) cannot be extended to \([0, 1] \) in such a way to make \( f \) an entire function.

4. (20 pts)
   a) (5 pts) Give a careful statement of the Schwarz Lemma.
   b) (15 pts) Prove that any analytic function \( f \) that maps the unit disc into itself, but is not one-to-one, must satisfy \( |f'(0)| < 1 \). (Note, we do NOT assume that \( f(0) = 0 \) here.)

5. (20 pts) Suppose \( f \) is analytic on a neighborhood of the closed unit disc. If \( |f(z)| < 1 \) when \( |z| = 1 \), prove that there must exist at least one point \( z \) with \( |z| < 1 \) such that \( f(z) = z \).