QUALIFYING EXAMINATION
AUGUST 2002
MATH 530 - Prof. Bell

1. *(20 pts)* Suppose that f is a continuous complex valued function on the unit disc $D_1(0)$ and that f is analytic on the upper half disc, $\{z : \text{Im } z > 0\} \cap D_1(0)$, and analytic on the lower half disc, $\{z : \text{Im } z < 0\} \cap D_1(0)$. Use only Morera's Theorem to prove that f is must actually be analytic on the whole disc.

2. *(20 pts)* Evaluate the integral $\displaystyle\int_0^\infty \sin(x^2)\, dx$ by integrating e^{iz^2} around the contour that starts at the origin and follows the real line out to a point $R > 0$, then follows the circular arc Re^{it} from $t = 0$ to $t = \pi/4$, and returns to the origin along the line joining $Re^{i\pi/4}$ to 0. Let $R \to \infty$. You may use the fact that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$ without proving it.

3. *(20 pts)* Prove that the integral

$$\int_0^1 \frac{t^2}{t-z}\, dt$$

defines an analytic function $f(z)$ on $\mathbb{C} - [0, 1]$. State Liouville's Theorem and use it to prove that f cannot be extended to $[0, 1]$ in such a way to make f an entire function.

4. *(20 pts)*

a) *(5 pts)* Give a careful statement of the Schwarz Lemma.
b) *(15 pts)* Prove that any analytic function f that maps the unit disc into itself, but is not one-to-one, must satisfy $|f'(0)| < 1$. (Note, we do NOT assume that $f(0) = 0$ here.)

5. *(20 pts)* Suppose f is analytic on a neighborhood of the closed unit disc. If $|f(z)| < 1$ when $|z| = 1$, prove that there must exist at least one point z with $|z| < 1$ such that $f(z) = z$.