QUALIFYING EXAMINATION

AUGUST 2004

MATH 530 - Prof. Bell

Each problem is worth 20 points

Notation: $D_r(a) = \{z : |z - a| < r\}$

1. How many zeros does the function $f(z) = e^z + z^{11} + 2004$ have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$? Explain.

2. State the Schwarz Lemma. Use it to prove that if f is a *one-to-one* analytic map of the unit disc *onto* itself such that $f(0) = 0$, then $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

3. Suppose that $f(z)$ is analytic in a neighborhood of the origin and

$$\sum_{n=1}^{\infty} f^{(n)}(0)$$

converges. Prove that $f(z)$ extends to be an entire function.

4. Let f be an entire function such that $|f(z)| \leq 2004 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.

5. Suppose that f is a non-constant analytic function on an open set Ω containing the closed unit disc. If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in D_1(0)$ such that $|f(z_0)| < 1$, show that $f(\Omega)$ contains $D_1(0)$.

6. Evaluate

$$\int_0^{\infty} \frac{\sin x}{x}\, dx$$

by complex variable methods.

7. Suppose that $f(z)$ is analytic on a simply connected domain Ω minus two points a_1 and a_2 in Ω. If the residue of f at a_1 is R_1 and the residue of f at a_2 is R_2, prove that there is an analytic function $F(z)$ on $\Omega - \{a_1, a_2\}$ such that

$$F'(z) = f(z) - \frac{R_1}{z - a_1} - \frac{R_2}{z - a_2}.$$