1. \(10\text{ pts}\) Suppose \(f\) and \(g\) are analytic functions on a domain \(\Omega\) and that \(f\) and \(g\) satisfy the identity
\[
f'(z) = g'(z)f(z)
\]
for all \(z\) on a closed line segment contained in \(\Omega\). Prove that \(f(z) = ce^{g(z)}\) on \(\Omega\) for some constant \(c\). You must explain your steps carefully.

2. \(30\text{ pts}\) Suppose that \(w_1, \ldots, w_N\) are points on the unit circle. Prove that there is a point \(z\) on the unit circle such that the product of the distances from \(z\) to the points \(w_j, j = 1, \ldots, N\) is exactly equal to one. Hint: Use an analytic function, not analytic geometry.

3. \(30\text{ pts}\) Suppose that \(f\) is an analytic function on the complex plane minus a discrete set of points \(A\). Suppose further that for each point \(a\) in \(A\), \(f\) has a simple pole with residue equal to a positive integer \(n(a)\). Let \(z_0\) denote a point in \(\mathbb{C} - A\). For a point \(z\) in \(\mathbb{C} - A\), let \(\gamma_z\) denote a contour in \(\mathbb{C} - A\) that starts at \(z_0\) and ends at \(z\).
   a) Prove that \(\mathbb{C} - A\) is connected.
   b) Prove that the formula
   \[
   F(z) = \exp \left( \int_{\gamma_z} f(w) \, dw \right)
   \]
   yields a well defined analytic function on \(\mathbb{C} - A\).
   c) Prove that \(F\) has a removable singularity at each point in \(A\).

4. \(30\text{ pts}\) Suppose \(f\) and \(g\) are analytic in a disc \(D_R(0)\) with \(R > 1\) and suppose that \(f\) has a simple zero at \(z = 0\) and has no other zeroes in the set \(\{z : |z| \leq 1\}\). Let
\[
H_\epsilon(z) = f(z) + \epsilon g(z).
\]
Prove that there is a radius \(r\) with \(0 < r < 1\) such that \(H_\epsilon(z)\) has a unique zero \(z_\epsilon\) in the unit disc if \(0 < \epsilon < r\).

Finally, prove that the mapping \(\epsilon \mapsto z_\epsilon\) is a continuous map from \((0, r)\) into the unit disc. Hint: What is the residue of \(\frac{z H_\epsilon'(z)}{H_\epsilon(z)}\) at \(z_\epsilon\)?