QUALIFYING EXAMINATION
JANUARY 2004
MATH 530 - Prof. Bell

1. *(10 pts)* Suppose f and g are analytic functions on a domain Ω and that f and g satisfy the identity

$$f'(z) = g'(z)f(z)$$

for all z on a closed line segment contained in Ω. Prove that $f(z) = ce^{g(z)}$ on Ω for some constant c. You must explain your steps carefully.

2. *(30 pts)* Suppose that $w_1, \ldots, w_N$ are points on the unit circle. Prove that there is a point z on the unit circle such that the product of the distances from z to the points w_j, $j = 1, \ldots, N$ is exactly equal to one. Hint: Use an analytic function, not analytic geometry.

3. *(30 pts)* Suppose that f is an analytic function on the complex plane minus a discrete set of points $\mathcal{A}$. Suppose further that for each pont a in $\mathcal{A}$, f has a simple pole with residue equal to a positive integer $n(a)$. Let z_0 denote a point in $\mathbb{C} - \mathcal{A}$. For a point z in $\mathbb{C} - \mathcal{A}$, let γ_z denote a contour in $\mathbb{C} - \mathcal{A}$ that starts at z_0 and ends at z.

a) Prove that $\mathbb{C} - \mathcal{A}$ is connected.

b) Prove that the formula

$$F(z) = exp\left(\int_{\gamma_z} f(w)\, dw\right)$$

yields a well defined analytic function on $\mathbb{C} - \mathcal{A}$.

c) Prove that F has a removable singularity at each point in $\mathcal{A}$.

4. *(30 pts)* Suppose f and g are analytic in a disc $D_R(0)$ with $R > 1$ and suppose that f has a simple zero at $z = 0$ and has no other zeroes in the set $\{z : |z| \le 1\}$. Let

$$H_\epsilon(z) = f(z) + \epsilon\, g(z).$$

Prove that there is a radius r with $0 < r < 1$ such that $H_\epsilon(z)$ has a unique zero z_ϵ in the unit disc if $0 < \epsilon < r$.

Finally, prove that the mapping $\epsilon \mapsto z_\epsilon$ is a continuous map from $(0, r)$ into the unit disc. Hint: What is the residue of $\frac{z\, H_\epsilon'(z)}{H_\epsilon(z)}$ at z_ϵ?