QUALIFYING EXAMINATION

JANUARY 2006

MATH 530 - Prof. Weitsman

1. (20) Find the radii of convergence of the following.

$$i)\ \sum_{n=0}^{\infty} \frac{1}{(1+2i)^n} z^n \qquad\qquad ii)\ \sum_{n=0}^{\infty} \frac{z^{n^2}}{n!}$$

2. (20) Show that

$$\int_{-\infty}^{\infty} \frac{x \sin(2x)}{x^4+16} dx = \frac{\pi e^{-2\sqrt{2}} \sin(2\sqrt{2})}{4}.$$

Show all work.

3. (20) Expand $f(z) = \dfrac{1}{z(z+2)}$ as a Laurent series

i) for $0 < |z| < 2$,

ii) for $|z-3| < 3$.

4. (20) Prove that all roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $\{z : 1 < |z| < 2\}$.

5. (20) Find all linear fractional transformations $T(z)$ which map the upper half plane $H^+ = \{z : \Im m z > 0\}$ onto the unit disk $U = \{z : |z| < 1\}$ such that $T(i) = 0$.

6. (20) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the unit disk $U = \{z : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Prove that if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is one-to-one in U.

7. (20) Suppose that $f(z)$ is analytic in the disk $\{z : |z| < R\}$ $(R > 1)$ except for a simple pole at $z = 1$ with residue -1. If its Taylor expansion in the unit disk is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

prove that $a_n \to 1$ as $n \to \infty$.