Math 530 Qualifying Exam, January 2008

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All problems have equal weight, and it is split equally between a) and b) if there are a) and b). In your proofs, you can use any theorem stated in class provided that you state it completely and correctly.

Name:

1. Suppose that a function u, harmonic in a neighborhood of the origin, equals to zero on the real and imaginary axes. Prove that u is even.

2. a) Prove that the series

$$f(z) = \sum_{n=0}^{\infty} 2^{-n^2} z^{2^n}$$

converges uniformly in the *closed* unit disc, and that the limit function is infinitely differentiable in the closed unit disc and analytic in the open unit disc.

b) Prove that this function f is not analytic at any point of the unit circle, that is the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is zero for every z_0 on the unit circle.

3. Evaluate the integral

$$\int_0^{\infty} \frac{\sin(\mathrm{Log} x)}{x^2 + 4} dx.$$

Here Log is the principal value of the logarithm (real on the positive ray).

4. Find a fractional linear transformation that maps the two circles

$$\{z : |z| = 1\} \quad \text{and} \quad \{z : |z - 1| = 3\}$$

onto some concentric circles.

5. a) Prove that every function of the form

$$az + b - \sum_{n=1}^{m} \frac{c_k}{z - t_k}, \tag{1}$$

where b and t_k are real, $a \geq 0$ and $c_k > 0$, maps the upper half-plane into itself and the lower half-plane into itself.

b) Prove that every rational function that maps the upper half-plane into itself and the lower half-plane into itself has the form (1).

6. Let f be an analytic function in a neighborhood of 0, that satisfies the functional equation

$$f'(z) = qf(q^2 z), \quad f(0) = 1,$$

with some q, $|q| \leq 1$. Find explicitly the Taylor coefficients of f at 0 and then determine the radius of convergence of its Taylor series.