MATH 530 Qualifying exam, January 2010, A. Eremenko

Notation: $\Re z$ and $\Im z$ are real and imaginary parts of z, $\mathbf{C}$ is the complex plane, "entire" means analytic in $\mathbf{C}$, "meromorphic function" means a ratio of two analytic functions. Each problem is worth 10 points.

1. Find a conformal map f of the upper half-plane onto the region

$$\{z \in \mathbf{C} : |z - 2i| < 2, |z - i| > 1\}$$

such that $f(0) = 2i, f(\pm 1) = 0$.

2. Let f be an analytic function in the closed unit disc which satisfies

$$|f(0)| + |f'(0)| < \inf\{|f(z)| : |z| = 1\}.$$

Prove that f has at least two zeros (counting multiplicity) in the open unit disc.

3. Let f and g be two entire functions, and $|f(z)| \leq |g(z)|$ for all complex z. Prove that $f = cg$, where c is a constant, or $g = 0$.

4. Let f be an analytic function in the unit disc satisfying $|\Re f(z)| < 1$ for all z in the unit disc. Prove that $|f'(0)| \leq 2$.

5. Let f and g be two analytic functions in a region D. Suppose that $f(z)\overline{g(z)}$ is real for all $z \in D$. Prove that $f = cg$ where c is a real constant, or $g \equiv 0$.

6. Let f be an analytic function in the region $\{z : 0 < |z| < 1\}$, and suppose that it satisfies

$$|f(z)| \leq A + B|z|^{-\alpha}$$

for some positive numbers A, B and α. Prove that f is meromorphic in the unit disc (that is 0 is either a pole or a removable singularity of f).

7. Let

$$f = \sum_{n=1}^{\infty} a_n z^n$$

be a conformal map from the unit disc onto the square $\{z : |\Re z| < 1, \ |\Im z| < 1\}$. Prove that $a_n = 0$ for all $n \neq 4k + 1, \ k = 0, 1, 2, 1 \ldots.$

8. Evaluate the integral

$$\int_0^\infty \frac{\log x}{1+x^2}.$$

Hint: integrate $\log z/(1+z^2)$ over the boundary of the region $\{z : \epsilon < |z| < R,\ \Im z > 0\}$.