MATH 530 Qualifying Exam

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1. (a) (10 points) Define

$$f(z) := \int_{-\pi}^{\pi} \frac{\cos^2 t}{z - \sin t}\, dt.$$

Use the difference quotient definition of complex derivative to show that f is holomorphic on $\mathbb{C} - [-1, 1]$ and to write $f'(z)$ as an integral.

(b) (10 points) Can $f(z)$ be extended to $[-1, 1]$ to make $f(z)$ an entire function? If so, describe how to extend f. If not, prove that no such extension exists.

2. (20 points) Let

$$P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Prove that for a given $R > 0$, there exists a positive integer N such that if $n \geq N$, then $P_n(z)$ has exactly n zeroes in $\{z : |z| > R\}$.

3. (20 points) Suppose that f has a pole at z_0 and that g is holomorphic on $\mathbb{C} - \overline{\mathbb{D}}$ (so that g has an isolated singularity at ∞). Give necessary and sufficient conditions on the singularity of g at ∞ so that $g(f(z))$ has a pole at z_0 and prove that your conditions are necessary and sufficient.

4. (a) (5 points) Suppose f has a pole of order m at z_0. Use the Laurent series expansion of f around z_0 to derive the formula for the residue of f at z_0 in terms of derivatives of $(z - z_0)^m f(z)$.

(b) (15 points) Compute

$$\int_0^\infty \frac{\ln x}{(x^2 + 1)^2} dx.$$

5. (20 points) Prove or disprove that there is a sequence of polynomials $\{p_n(z)\}_{n=1}^\infty$ so that $p_n(z)$ converges uniformly on the unit circle, $\{z : |z| = 1\}$, to the function $f(z) = (\overline{z})^2$.