1. Define \( f(z) = \int_0^1 \frac{dt}{1 + tz} \).

   (a) (10 points) Use Morera’s Theorem to show that \( f \) is analytic in \( \mathbb{D} \).

   (b) (10 points) Find a power series expansion for \( f(z) \) on \( \mathbb{D} \).

2. (20 points) Suppose \( f \) is entire and that there is some \( K > 0 \) so that if \( |z| \geq K \), then

   \[ |\text{Re} f(z)| \geq |\text{Im} f(z)|. \]

   Prove that \( f \) is constant.

3. (20 points) Suppose \( f \) is a holomorphic function (not necessarily bounded) on \( \mathbb{D} \) such that \( f(0) = 0 \). Prove that the infinite series \( \sum_{n=1}^{\infty} f(z^n) \) converges uniformly on compact subsets of \( \mathbb{D} \).

4. (20 points) Define a family \( \mathcal{F} \) of functions holomorphic in an open set \( \Omega \) to be a normal family if every sequence from \( \mathcal{F} \) contains either a subsequence that converges uniformly on every compact set \( K \subset \Omega \) or a subsequence that tends uniformly to \( \infty \) on every compact set.

   Let \( f \) be entire, and let \( \mathcal{F} \) be the family \( \{ f(kz) : k \in \mathbb{C} \} \). Also, let \( \Omega \) be the annulus \( 1/2 < |z| < 2 \). Prove that \( \mathcal{F} \) is a normal family in \( \Omega \) if and only if \( f \) is a polynomial.

5. Let \( \Omega \) be a bounded, open, connected set, and suppose that \( f_1, f_2, \ldots, f_n \) are holomorphic in \( \Omega \) and continuous on the closure of \( \Omega \). Let \( g = |f_1| + |f_2| + \cdots + |f_n| \).

   (a) (10 points) Prove that the maximum of \( g \) is attained on the boundary of \( \Omega \).

   (b) (10 points) Prove that if \( g \) is constant, then each \( f_k \) is also constant.