MATH 530 Qualifying Exam

January 2012 (S. Bell)

Each problem is worth 20 points

1. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\, dx.$$

Hint: Integrate $f(z) = \dfrac{e^{iz}}{e^z + e^{-z}}$ around the boundary of the rectangle with vertices at $\pm R$, $\pm R + i\pi$ and let $R \to \infty$.

2. Suppose that u is a real valued harmonic function on the unit disc. Show that there is a sequence of real valued *harmonic polynomials* that converges uniformly on compact subsets of the unit disc to u. Given the real valued harmonic function $u = \ln |z|$ on $\mathcal{A} = \{z : 1 < |z| < 2\}$, is it possible to find a sequence of harmonic polynomials which converges uniformly on compact subsets of $\mathcal{A}$ to u? Explain.

3. Let $\mathcal{F}$ denote the family of all analytic functions f on the unit disc that map the unit disc into itself with $f(1/2) = 0$. Find $\sup \{\text{Im } f(0) \ : \ f \in \mathcal{F}\}$.

4. a) State Rouché's Theorem for the unit disc.

b) Use Rouché's Theorem to prove that a polynomial of degree $N \geq 1$ has N roots (counted with multiplicities) in the complex plane.

5. Prove that

$$e^z + \frac{1}{(z-1)^3} + \frac{1}{(z+1)^6}$$

has an analytic cube root on $\mathbb{C} - \{\pm 1\}$.