1. Suppose $\Omega$ is a domain in the complex plane and $F(z, t)$ is a continuous function on $\Omega \times I$ where $I = [0, 1]$ is the unit interval in $\mathbb{R}$. Suppose further that $F(z, t)$ is analytic in $z$ on $\Omega$ for each fixed $t \in I$. Prove that

$$g(z) = \int_0^1 F(z, t) \, dt$$

is analytic on $\Omega$. What can be said if $F(z, t)$ is only assumed to be analytic in $z \in \Omega$ for all rational values of $t$ (when held fixed) in $I$.

2. Let $C_1$ denote the unit circle parametrized in the standard sense. Compute

$$\int_{C_1} \frac{1}{z^2 + z - \sigma} \, dz$$

where $\sigma$ is a real number satisfying $0 < \sigma < 2$.

3. Suppose that $f(z)$ is analytic on the upper half plane and maps the upper half plane into the unit disc. Prove that $|f'(i)| \leq \frac{1}{2}$. What can be said if $|f'(i)| = \frac{1}{2}$?

4. Suppose $f$ is analytic on a domain $\Omega$ and is not identically zero there. Let $Z_f = \{z \in \Omega : f(z) = 0\}$ denote the zero set of $f$. Prove that $\Omega - Z_f$ is connected. Is the same true if $f$ is assumed to be a real valued harmonic function?

5. Suppose $a_n$ is a sequence of distinct non-zero complex numbers such that

$$\sum_{n=1}^{\infty} |a_n|^{-1} < \infty.$$ 

Let $A = \{a_n : n = 1, \ldots, \infty\}$.

a) Prove that $\sum_{n=1}^{\infty} \frac{1}{z - a_n}$ converges to a function $f(z)$ that is analytic on $\mathbb{C} - A$.

b) For $z \in \mathbb{C} - A$, let

$$G(z) = \exp \left( \int_{\gamma_0^z} f(w) \, dw \right)$$

where $\gamma_0^z$ is a curve in $\mathbb{C} - A$ that starts at the origin and ends at $z$. Prove that $G$ is well defined and analytic on $\mathbb{C} - A$. Show that $G$ has removable singularities at each of the points $a_n$. Finally, show that the points $a_n$ are in fact simple zeroes of $G$. 