**MATH 530 Qualifying Exam**

January 2014 (S. Bell)

*Each problem is worth 20 points*

*Notation:* $D_r(a)$ *denotes the open disc of radius* $r$ *about* $a$.

**1.** Suppose $\varphi$ is a continuous function on a path $\gamma$ and $D_r(a)$ is a disc whose closure does not intersect $\gamma$. Suppose that $z(t)$, $\alpha \le t \le \beta$, is a continuous, piecewise $C^1$ parameterization of $\gamma$. Let $L$ denote the length of $\gamma$,
$d = \inf\{|z(t) - a| : \alpha \le t \le \beta\}$ denote the distance from $a$ to $\gamma$, and
$M = \sup\{|\varphi(z(t))| : \alpha \le t \le \beta\}$. Suppose further that $w \in D_r(a)$.

Carefully bound the integral $\displaystyle\int_\gamma \frac{\varphi(z)}{(z-w)^2(z-a)}\, dz$ in terms of $M$, $d$, $r$, and $L$.

**2.** Suppose that $a_1, a_2, \ldots, a_N$ are distinct points in the complex plane contained in a circle of radius $R_0$ where $N \ge 4$. Let $Q(z)$ denote the rational function given by

$$Q(z) = \frac{z^2}{\prod_{n=1}^N (z - a_n)}.$$

a) What is the residue of $Q(z)$ at one of the points $a_k$?

b) State a version of the Residue Theorem that is most relevant to computing $\int_{C_R} Q(z)\, dz$, where $C_R$ denotes a circle of radius $R > R_0$ about the origin parameterized in the counterclockwise sense.

c) State the most general Residue Theorem you know.

d) Prove that the integral in part (b) tends to zero as $R \to \infty$.

**3.** Suppose that $f(z)$ is analytic on the unit disc and maps the unit disc into itself. If $a$ is a point in the unit disc, how big could $|f'(a)|$ be? Explain.

**4.** Suppose that $m$ and $n$ are positive integers with $n > m$. Find the point or points in the closed unit disc where $|z^n - z^m|$ assumes its maximum value. Find the point or points where it assumes its minimum value.

**5.** Show that if $\varphi$ is a real valued harmonic function on the unit disc that is continuous up to the boundary such that $\varphi$ agrees with a real valued polynomial on the unit circle, then $\varphi$ must be a harmonic real valued polynomial.

Hints: A real valued polynomial in $x$ and $y$ can be rewritten as a polynomial in $z = x + iy$ and $\bar{z} = x - iy$ via $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. Note that $\bar{z} = 1/z$ on the unit circle. Note also that $z = 1/\bar{z}$ there too.