REAL ANALYSIS
QUALIFYING EXAM
JANUARY 2018

Observation. **You have 2 hours to complete this exam. Books, notebooks, and any other course materials are NOT allowed. Cell phones must be turned off. No computers or calculators are accepted. Each problem should be solved on a distinct (new) page (if you need more space ask for supplementary paper).**

Problem	Points
1	
2	
3	
4	
Total	

Ex.1 (20 points) Let $\mathbb{T}$ be the standard one dimensional torus (unit circle) given by

$$\mathbb{T} := \left\{ e^{2\pi i x} \,|\, x \in [-\frac{1}{2}, \frac{1}{2}] \right\} .$$

We consider on $\mathbb{T}$ the standard Lebesgue measure m via the natural identification of $\mathbb{T}$ with the unit interval centered at the origin. Further, given $E \subseteq \mathbb{T}$ Lebsesgue measurable and $t \in [-\frac{1}{2}, \frac{1}{2}]$ we denote with E_t the t-translate of E, that is

$$E_t := \{ e^{2\pi i (x+t)} \,|\, e^{2\pi i x} \in E \} .$$

Prove the following:

a) If E, $F \subseteq \mathbb{T}$ Lebesgue measurable then there exists a translate F_t such that

$$m(E \cap F_t) = m(E)\, m(F) .$$

<u>Hint</u>: *Express $m(E \cap F_t)$ in an integral form and then study the properties of the function $t \to m(E \cap F_t)$.*

b) If $E \subseteq \mathbb{T}$ Lebesgue measurable with $m(E) > 0$ then there exist n translates of E whose union has measure exceeding $\frac{1}{2}$ provided $n > \frac{\ln 2}{m(E)}$.

<u>Hint</u>: *If $E := \bigcup_{j=1}^{n} E_{t_j}$ and $F_{t_j} := (E_{t_j})^c$ apply successively a) for the set $\bigcap_{j=1}^{n} F_{t_j}$.*

<u>Present in great detail all your reasonings.</u>

Solution Ex.1 (continuation)

Ex.2 (25 points) i) Construct a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : [0,1] \to [0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0\,, \tag{1}$$

but such that the sequence $\{f_n(x)\}_n$ converges for no $x \in [0,1]$.

Deduce thus that convergence in norm does not imply a.e. pointwise convergence.

ii) Show that in the setting described by i) one can always extract a subsequence $\{f_{n_k}\}_k$ which converges (Lebesgue) almost everywhere at $f \equiv 0$.

iii) How about the following partial reverse implication: is it true that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions with $f_n : [0,1] \to [0,1]$ and such that

$$\exists \lim_{n\to\infty} f_n(x) = 0 \quad \text{a.e. } x \in [0,1]\,,$$

then (1) must hold? Justify your answer.

Present in great detail all your reasonings.

Solution Ex.2 (continuation)

Ex.3 (25 points) Let $\{K_n\}_{n\in\mathbb{N}}$ be a family of functions in[1] $L^1(\mathbb{T})$ that forms a family of good kernels. More precisely, that means that all of the following properties are satisfied:

(2)

- $\int_{[-\frac{1}{2},\frac{1}{2}]} K_n(x)\,dx = 1$ uniformly in $n \in \mathbb{N}$.
- there exits $M > 0$ such that $\int_{[-\frac{1}{2},\frac{1}{2}]} |K_n(x)|\,dx \le M$ for any $n \in \mathbb{N}$.
- for every $\eta \in (0, \frac{1}{2})$ one has

$$\lim_{n\to\infty} \int_{\eta<|x|\le\frac{1}{2}} |K_n(x)|\,dx = 0\,.$$

i) Prove that if $\{K_n\}_{n\in\mathbb{N}}$ is a family of good kernels then for any $g \in C(\mathbb{T})$ one has[2]

$$\exists \lim_{n\to\infty} \|g - g * K_n\|_{C(\mathbb{T})} = 0\,. \tag{3}$$

ii) Let[3] $\{D_n\}_{n\in\mathbb{N}}$ be a family of functions with the general term given by

$$D_n(x) := \sum_{j=-n}^{n} e^{2\pi i j x}\,. \tag{4}$$

Prove that the Dirichlet kernel D_n can be re-written as

$$D_n(x) := \frac{\sin((2n+1)\,\pi\,x)}{\sin(\pi x)}\,,$$

and further deduce that $\{D_n\}_{n\in\mathbb{N}}$ *is not* a family of good kernels.

iii) Let $\{K_n\}_{n\in\mathbb{N}^*}$ be a family of functions with the general term given by

$$K_n(x) := \frac{D_0(x) + \ldots + D_{n-1}(x)}{n}\,. \tag{5}$$

The Fejer kernel K_n can be re-written as[4]

$$K_n(x) := \frac{1}{n}\,\frac{\sin^2(n\,\pi\,x)}{\sin^2(\pi x)}\,. \tag{6}$$

Prove that $\{K_n\}_{n\in\mathbb{N}^*}$ *is* a family of good kernels.

Present in great detail all your reasonings.

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[1] Recall the definition of the torus $\mathbb{T}$ and its identification with the interval $[-\frac{1}{2}, \frac{1}{2}]$. The class $L^1(\mathbb{T})$ here stands for the set of 1−periodic functions $f : \mathbb{R} \to \mathbb{C}$ which are Lebesque measurable on $[-\frac{1}{2}, \frac{1}{2}]$ and which obey the condition $\int_{[-\frac{1}{2},\frac{1}{2}]} |f(x)|\,dx < \infty$.

[2] Here the class $C(\mathbb{T})$ designates the class of continuous (periodic) functions on the torus endowed with the standard norm $\|g\|_{C(\mathbb{T})} := \sup_{x\in\mathbb{T}} |g(x)|$.

[3] Throughout this exam we use the convention that the set of natural numbers $\mathbb{N} := \{0, 1, \ldots\}$ while $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

[4] You do not need to prove formula (6). However you are invited to use both (5) and (6) in approaching point iii).

Solution Ex.3 (continuation)

Ex.4 (30 points) 1) Let $f \in L^1(\mathbb{T})$. Assume that the following relation holds:

$$\int_{\mathbb{T}} f(x)\,\overline{g(x)}\,dx = 0 \quad \text{for any} \quad g \in C(\mathbb{T})\,. \tag{7}$$

Prove that $f = 0$ a.e. $x \in \mathbb{T}$.

2) Prove the following much deeper and stronger[5] version of 1) above: assume that one has

$$\hat{f}(n) := \int_{\mathbb{T}} f(x)\,e^{-2\pi i n x}\,dx = 0 \quad \text{for any} \quad n \in \mathbb{Z}\,. \tag{8}$$

Then $f = 0$ a.e. $x \in \mathbb{T}$.

In order to prove this fact one is invited to complete the following steps:

Step 1. If $u,\, v \in L^1(\mathbb{T})$ then $\widehat{u * v}(n) = \hat{u}(n)\,\hat{v}(n)$ for any $n \in \mathbb{Z}$.

Step 2. If K_n, $n \in \mathbb{N}^*$ stands for the Fejer kernel defined in the previous exercise, see (5), show that that the sequence of Cesaro partial sums attached to an arbitrary $g \in C(\mathbb{T})$ and defined by

$$\sigma_n(g)(x) := (g * K_n)(x) \quad \text{for } n \in \mathbb{N}^*\,,$$

verifies

$$\sigma_n(g)(x) = \sum_{|j| \leq n} \left(1 - \frac{|j|}{n}\right) \hat{g}(j)\,e^{2\pi i j x}\,, \tag{9}$$

and hence

$$\int_{\mathbb{T}} f(x)\,\overline{\sigma_n(g)(x)}\,dx = 0 \quad \text{for any} \quad n \in \mathbb{N}^* \text{ and } g \in C(\mathbb{T})\,. \tag{10}$$

Step 3. Using ex. 3 i) and iii) and (10) deduce that $f = 0$ a.e. $x \in \mathbb{T}$.

3) Assume that we are given $f \in C(\mathbb{T})$ and $C = C(f) > 0$ depending only on f such that

$$|\hat{f}(n)| \leq \frac{C}{1 + |n|^{\frac{5}{2}}} \qquad \forall\, n \in \mathbb{Z}\,. \tag{11}$$

Show that the expression

$$h_N(x) := \sum_{|n| \leq N} \hat{f}(n)\,e^{2\pi i n x}\,,$$

converges uniformly (as $N \to \infty$) to an absolutely continuous function h on $[0,1]$ and show that $h = f$. Deduce that f is differentiable almost everywhere with $f' \in L^1(\mathbb{T})$ and prove that $\widehat{f'}(n) = 2\pi i n \hat{f}(n)$.

Present in great detail all your reasonings.

[5] This is often referred as the uniqueness of Fourier expansion.

Solution Ex.4 (continuation)