MATH 544 QUALIFYING EXAMINATION
January 2017

Student Identifier: 

(PLEASE PRINT CLEARLY)

Instructions: This exam consists of 6 problems. A problem appears on each of the following pages. Use the space provided for the solutions, using back pages as needed.
Problem 1. (10 pts) Let $f$ be a non-negative measurable function on a finite measure space $(X, \mathcal{F}, \mu)$. Prove that the sequence $\int_X f^n d\mu, \ n = 1, 2, \ldots,$ tends either to $+\infty$ or to $\mu\{x \in X : f(x) = 1\}$. 
Problem 2. (10–pts) Consider $[0, 1]$ with its Lebesgue measure. Let $1 < p < \infty$ and set

$$\Gamma = \left\{ f \in L^p([0, 1]) : \int_0^1 5f(x)x^3dx \leq \frac{1}{\pi} \int_0^1 f(x)dx \right\}$$

(a subset of the metric space $M = L^p([0, 1])$ with the metric $d_M(f, g) = \|f - g\|_p$). Prove that $\Gamma$ is closed in $L^p([0, 1])$ (equivalently its complement is open).
Problem 3. (10–pts) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $E_1, \ldots, E_{50}$ be measurable sets with the property that almost every $x \in X$ belongs to at least 10 of these sets. Prove that at least one of these sets must have measure greater than or equal to $1/5$. 
**Problem 4. (10–pts)** Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Prove that the following limit

$$\lim_{m \to \infty} \int_0^{\infty} \frac{e^{-x} f(x + 2)}{2^m x^2 + 2^{-m}} \, dx$$

exists and find it.
Problem 5. Let $f_n$ be sequence of function on $[0,1]$ with each $f_n$ increasing. That is if $x, y \in [0,1]$ with $x \leq y$, then $f_n(x) \leq f_n(y)$ for every $n$. Suppose $(f_n(1) - f_n(0)) \leq 25$ for all $n$.

(i) (10–pts) Prove that for every $\beta > 0$,

$$ \liminf_{n \to \infty} \left\{ \frac{f'_n(x)}{n^\beta} \right\} = 0, \text{ a.e.} $$

(ii) (10–pts) Prove that for every $\alpha > 1$,

$$ \lim_{n \to \infty} \left\{ \frac{f'_n(x)}{n^\alpha} \right\} = 0, \text{ a.e.} $$
Problem 6. (10–pts) Let \( f : [0, 1] \rightarrow \mathbb{R} \) be absolutely continuous. Suppose \( f' \in L^p([0, 1]) \), \( 1 < p < \infty \). Let \( q \) be the conjugate exponent of \( p \). That is, \( \frac{1}{q} + \frac{1}{p} = 1 \). Set \( \beta = \frac{1}{q} \).

(i) (10–pts) Prove that \( f \) is Hölder continuous of order \( \beta \).

(ii) (10–pts) Prove that

\[
\lim_{h \to 0^+} \frac{|f(a + h) - f(a)|}{h^\beta} = 0, \quad a \in (0, 1).
\]