MATH 544 QUALIFYING EXAMINATION
January 2017

Student Identifier: ____

(PLEASE PRINT CLEARLY)

Instructions: This exam consists of 6 problems. A problem appears on each of the following pages. Use the space provided for the solutions, using back pages as needed.

Problem 1. (10 pts) Let f be a non-negative measurable function on a finite measure space $(X, \mathcal{F}, \mu)$. Prove that the sequence $\int_X f^n d\mu$, $n = 1, 2, \ldots$, tends either to $+\infty$ or to $\mu\{x \in X : f(x) = 1\}$.

Problem 2. (10–pts) Consider $[0,1]$ with its Lebesgue measure. Let $1 < p < \infty$ and set

$$\Gamma = \left\{ f \in L^p([0,1]) : \int_0^1 5f(x)x^3 dx \leq \frac{1}{\pi} \int_0^1 f(x)dx \right\}$$

(a subset of the metric space $M = L^p([0,1])$ with the metric $d_M(f,g) = \|f - g\|_p$). Prove that Γ is closed in $L^p([0,1]$ (equivalently its complement is open).

Problem 3. **(10–pts)** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $E_1, \ldots, E_{50}$ be measurable sets with the property that almost every $x \in X$ belongs to at least 10 of these sets. Prove that at least one of these sets must have measure greater than or equal to $1/5$.

Problem 4. **(10–pts)** Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Prove that the following limit

$$\lim_{m\to\infty} \int_0^\infty \frac{e^{-x} f(x+2)}{2^m x^2 + 2^{-m}} dx$$

exists and find it.

Problem 5. Let f_n be sequence of function on $[0,1]$ with each f_n increasing. That is if $x, y \in [0,1]$ with $x \leq y$, then $f_n(x) \leq f_n(y)$ for every n. Suppose $(f_n(1) - f_n(0)) \leq 25$ for all n.

(i) **(10–pts)** Prove that for every $\beta > 0$,

$$\liminf_{n\to\infty} \left\{ \frac{f_n'(x)}{n^\beta} \right\} = 0, \quad a.e.$$

(ii) **(10–pts)** Prove that for every $\alpha > 1$,

$$\lim_{n\to\infty} \left\{ \frac{f_n'(x)}{n^\alpha} \right\} = 0, \quad a.e.$$

Problem 6. **(10–pts)** Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Suppose $f' \in L^p([0,1])$, $1 < p < \infty$. Let q be the conjugate exponent of p. That is, $\frac{1}{q} + \frac{1}{p} = 1$. Set $\beta = \frac{1}{q}$.

(i) **(10–pts)** Prove that f is Hölder continuous of order β.

(ii) **(10–pts)** Prove that

$$\lim_{h \to 0+} \frac{|f(a+h) - f(a)|}{h^\beta} = 0, \qquad a \in (0,1).$$