1. Assume $f_n$ is a measurable function on $\mathbb{R}$ for $n = 1, 2, \ldots$, and $f_n(x) \to f(x), \forall x$. In each case say whether the additional hypotheses given imply that $\int f_n(x) dx \to \int f(x) dx$ and justify your answer.

(5) a. $\forall n$, $|f_n| \leq 1$ and $m(\{x : f_n(x) \neq 0\}) \leq 1$.

(5) b. $\forall n, \forall x$, $|f_n(x)| \leq \frac{1}{1 + x^2}$.

(5) c. $\forall n$, $f_n \geq 0$ and $\int f_n(x) dx \leq 1$.

(5) d. $\forall n$, $0 \leq f_n \leq f_{n+1}$ and $\int f_n(x) dx \leq 1$.

2. Assume $f_n$ is a measurable function on $[0, 1]$ for $n = 1, 2, \ldots$, $|f_n| \leq g$, $\forall n$, $\int_0^1 g(x) dx < \infty$, and $F_n(x) = \int_0^x f_n(t) dt$ for $x$ in $[0, 1]$. Show that $(F_n)_{n=1}^\infty$ has a uniformly convergent subsequence.

3. Let $f$ be a measurable function on a measure space $(S, \mu)$, and assume $1 \leq p_1 < p < p_2 < \infty$.

(7) a. Show that if $\|f\|_{p_1} < \infty$ and $\|f\|_{p_2} < \infty$, then $\|f\|_p < \infty$.

(8) b. Show that if $\mu(S) < \infty$ and $\|f\|_p < \infty$, then $\|f\|_{p_1} < \infty$.

(10) c. Show that there is a function $f$ on $[0, 1]$ with Lebesgue measure such that $\|f\|_1 < \infty$ and $\|f\|_p = \infty$, $\forall p > 1$.

4. For $f$ a real-valued function on $[0, 1]$ let $f_h(x) = \begin{cases} f(x + h), & x + h \in [0, 1] \\ 0, & x + h \notin [0, 1] \end{cases}$.

(10) a. Assume $f \in L^p$ and $1 \leq p < \infty$. Show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow \|f_h - f\|_p < \epsilon$.

(7) b. Assume $f$ is continuous and $f(0) = f(1) = 0$. Show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow \|f_h - f\|_\infty < \epsilon$.

(8) c. Prove the converse to b: If $f \in L^\infty$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow \|f_h - f\|_\infty < \epsilon$, then there is a continuous function $\tilde{f}$ such that $\tilde{f}(0) = \tilde{f}(1) = 0$.
and \( \tilde{f} = f \) almost everywhere. (Hint for c: First show that \( \forall \epsilon > 0 \exists \) a continuous function \( g_\epsilon \) such that \( ||f - g_\epsilon||_\infty < \epsilon \).)

5. Assume \( f \) is a real-valued function on \([0, 1]\) and
   (i) \( f \) is continuous from the right at each \( x \) in \([0, 1]\)
   (ii) The left-hand limit, \( \lim_{y \to x^-} f(y) \), exists for each \( x \) in \((0, 1]\).

(5) a. Show that \( f \) is bounded.

(10) b. Show that for each \( \epsilon > 0 \), there is a partition, \( 0 = x_0 < x_1 < \cdots < x_n = 1 \), such that whenever \( 0 \leq i < n \) and \( s, t \in [x_i, x_{i+1}) \), then \( |f(s) - f(t)| < \epsilon \).

(Note: Hypothesis (ii) cannot be dropped and the conclusion of b would be false if \([x_i, x_{i+1})\) is replaced by \([x_i, x_{i+1}]\).)