QUALIFYING EXAMINATION
MA 544

SPRING 1995

Name: ________________________________

Instructions. Standard notation is used throughout. In particular, \( \mathbb{R} \) = \{reals\}, \( \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} \), \( I \) is a compact interval in \( \mathbb{R} \), and \( |A| \) is the Lebesgue measure of \( A \), a measurable subset of \( \mathbb{R} \).

There will be 6 additional pages with a problem on each page. Use the space provided for your solution of the problem.
1. Let \( \{f_n\} \subset C(I) \) such that for \( x \in I, f_1(x) \leq f_2(x) \leq \cdots \rightarrow f(x) \) pointwise on \( I \). Show that \( \{f_n\} \) is equicontinuous on \( I \) if and only if \( f \in C(I) \).
2. Let \( f : \mathbb{R} \to \mathbb{R}_+ \) be measurable, and let \( 0 < r < \infty \). Show that

\[
\frac{1}{|I|} \int_I f \leq \left( \frac{1}{|I|} \int_I \frac{1}{f^r} \right)^{1/r}
\]

for every \( I \subset \mathbb{R} \).

(Hint: \(|I| = \int_I f^r f^{-r} \).)
3. Let \( \{f_n\} \) be a sequence of non-negative measurable functions in \( L^p(\mathbb{R}) \) for some \( 1 < p < \infty \). Show that \( f_n \to f \) if and only if \( f_n^p \to f^p \).
4. Let $f : \mathbb{R} \to \mathbb{R}_+$ be measurable, and let $\epsilon > 0$. Show that there exists $g : \mathbb{R} \to \mathbb{R}_+$ measurable such that (i) $\|f - g\|_\infty \leq \epsilon$ and (ii) for every $r \in \mathbb{R}$, $|\{x : g(x) = r\}| = 0$. 
5. Assume that $f \in AC(I)$ for every $I \subset \mathbb{R}$. If both $f$ and $f'$ are in $L^1(\mathbb{R})$ show that

(i) $\int_{\mathbb{R}} f' = 0$, and (ii) $f(x) \to 0$ as $|x| \to \infty$. 

6. For \( f : I \to \mathbb{R} \) let

\[
\overline{D} f(x) = \lim_{h \to 0} \sup h \frac{f(x + h) - f(x)}{h},
\]

\[
\underline{D} f(x) = \lim_{h \to 0} \inf h \frac{f(x + h) - f(x)}{h}.
\]

If \(-K \leq \overline{D} f(x) \leq \underline{D} f(x) \leq K < \infty\) for every \( x \in I \), show that \(|f(x') - f(x'')| \leq K|x' - x''|\) for every \( x', x'' \in I \).