Name: ________________________________

**Instructions.** Standard notation is used throughout. In particular, \( \mathbb{R} = \{ \text{reals} \} \), \( I_0 = [0, 1] \), and \( C(I_0), BV(I_0), AC(I_0), L^p(I_0) \) are the common function spaces over \( I_0 \). For a measurable subset \( A \) of \( \mathbb{R} \), let \( |A| \) denote the Lebesgue measure of \( A \). All functions are assumed to be measurable.

There will be 6 additional pages with a problem on each page. Use the space provided for your solution of the problem.
1. Let \( f \in L^1(I_0), f \geq 0 \), and let for each positive integer \( n \)
\[
    f_n(x) = \begin{cases} 
        n, & f(x) \geq n \\
        f(x), & f(x) < n.
    \end{cases}
\]

Show that
\[
    \int_0^1 \log f_n \, dx \to \int_0^1 \log f \, dx.
\]

Note that the integrals could be \(-\infty\).
2. Assume that $f_n \in L^p(I_0)$ for some $1 < p < \infty$ with $\|f_n\|_p \leq M < \infty$, $n = 1, 2, \ldots$. If $F_n(t) = \int_0^t f_n(x) \, dx$, show that there exists a subsequence $n_1 < n_2 < \cdots$ such that $\{F_{n_j}\}$ converges uniformly on $I_0$ to an absolutely continuous function $F$. 

3. Assume that with each \( x \in \mathbb{R} \) there are associated sequences \( \{x'_n\}, \{x''_n\} \) and \( 0 < c_x < \infty \) such that (i) \( x'_n > x''_n > x \), (ii) \( x'_n \to x \), (iii) \( (x'_n - x''_n)/(x'_n - x) \geq c_x \). If \( f \in L^1(\mathbb{R}) \), show that

\[
\frac{1}{x'_n - x''_n} \int_{x''_n}^{x'_n} f(t) \, dt \to f(x), \text{ a.e.} x.
\]

Give an example showing that (iii) can not be omitted. (Hint: Let \( f = \chi_C \), where \( C = ? \))
4. In this problem you may use without proof the fact that if $f \in L^1(\mathbb{R})$ and $f_t(x) = f(x-t)$, then $f_t \to f(L^1)$ as $t \to 0$. Let $A$ be a measurable subset of $\mathbb{R}$ with $0 < |A| < \infty$. Show that there exists $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$ there are points $x, y \in A$ with $|x - y| = \epsilon$.

(Hint: $|A \cap (A + \epsilon)| = \int_A \chi_A(x) \, dx \to ?$ as $\epsilon \to 0$.)
5. Let \((X, \mathcal{M}, \mu)\) be a measure space with \(\mu(X) < \infty\). Assume that \(f_n \to f, \mu - a.e.\) and that \(\sup_n \int_X |f_n|^{p_0} \, d\mu < \infty\) for some \(1 < p_0 < \infty\). Show that \(f_n \to f(L^1)\).
6. This problem is designed to test your intuition. Let \( f \in L^p(\mathbb{R}) \) for some \( 1 < p < \infty \), and let for each positive integer \( n \)

\[
L_n = \| f(x + 2n) - f((x + n)) \|_p, K_n = \| f(x + 2n) + f(x + n) \|_p.
\]

It is known that \( L_n \to L, K_n \to K \) as \( n \to \infty \). Which of the following two statements is true?

(i) \( L \neq K \) is possible, (ii) \( L = K \) always.

Give a one sentence explanation for your choice.