QUALIFYING EXAMINATION

MA 544

FALL 1996

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $I_0 = [0,1]$, and $C(I_0), BV(I_0), AC(I_0), L^p(I_0)$ are the common function spaces over I_0. For a measurable subset A of $\mathbb{R}$, let $|A|$ denote the Lebesgue measure of A. All functions are assumed to be measurable.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $f \in L^1(I_0)$, $f \geq 0$, and let for each positive integer n

$$f_n(x) = \begin{cases} n, & f(x) \geq n \\ f(x), & f(x) < n. \end{cases}$$

Show that

$$\int_0^1 \log f_n \, dx \to \int_0^1 \log f \, dx.$$

Note that the integrals could be $-\infty$.

2. Assume that $f_n \in L^p(I_0)$ for some $1 < p < \infty$ with $||f_n||_p \leq M < \infty, n = 1, 2, \cdots$. If $F_n(t) = \int_0^t f_n(x)\,dx$, show that there exists a subsequence $n_1 < n_2 < \cdots$ such that $\{F_{n_j}\}$ converges uniformly on I_0 to an absolutely continuous function F.

3. Assume that with each $x \in \mathbb{R}$ there are associated sequences $\{x'_n\}, \{x''_n\}$ and $0 < c_x < \infty$ such that (i) $x'_n > x''_n > x$, (ii) $x'_n \to x$, (iii)$(x'_n - x''_n)/(x'_n - x) \geq c_x$. If $f \in L^1(\mathbb{R})$, show that

$$\frac{1}{x'_n - x''_n} \int_{x''_n}^{x'_n} f(t)\, dt \to f(x), a.e.x.$$

Give an example showing that (iii) can not be omitted.(Hint: Let $f = \chi_C$, where $C =$?)

4. In this problem you may use without proof the fact that if $f \in L^1(\mathbb{R})$ and $f_t(x) = f(x-t)$, then $f_t \to f(L^1)$ as $t \to 0$. Let A be a measurable subset of $\mathbb{R}$ with $0 < |A| < \infty$. Show that there exists $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$ there are points $x, y \in A$ with $|x-y| = \epsilon$.

   (Hint: $|A \cap (A+\epsilon)| = \int_A \chi_A(?)\, dx \to ?$ as $\epsilon \to 0$.)

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Assume that $f_n \to f, \mu - a.e.$ and that $\sup_n \int_X |f_n|^{p_0}\, d\mu < \infty$ for some $1 < p_0 < \infty$. Show that $f_n \to f (L^1)$.

6. This problem is designed to test your *intuition*. Let $f \in L^p(\mathbb{R})$ for some $1 < p < \infty$, and let for each positive integer n

$$L_n = ||f(x+2n) - f((x+n)||_p, K_n = ||f(x+2n) + f(x+n)||_p.$$

It is known that $L_n \to L, K_n \to K$ as $n \to \infty$. Which of the following two statements is true?
(i) $L \neq K$ is possible, (ii) $L = K$ always.
Give a *one sentence* explanation for your choice.