Name: ____________________________

**Instructions.** Standard notation is used throughout. In particular, \( \mathbb{R} = \{ \text{reals} \} \), \( I_0 = [0, 1] \), and \( C(I_0), BV(I_0), AC(I_0), L^p(I_0) \) are the common function spaces over \( I_0 \). For a measurable subset \( A \) of \( \mathbb{R} \), let \( |A| \) denote the Lebesgue measure of \( A \). All functions are assumed to be measurable. If \( 1 \leq p \leq \infty \), then \( p' \) is the conjugate index, i.e., \( 1/p + 1/p' = 1 \).

There will be 6 additional pages with a problem on each page. Use the space provided for your solution of the problem.
1. Let \( f \in C(I_0) \). Show that there exists a sequence of polynomials \( \{p_n\} \) with integer coefficients such that \( p_n \) converges point-wise on \( I_0 \) to

\[
g(x) = \begin{cases} 
  f(x), & 0 < x < 1 \\
  0, & x = 0, 1.
\end{cases}
\]

(Hint: You may use without proof the fact that if \( f \in C(I_0) \), then such a sequence of polynomials \( \{p_n\} \) exists which converges on \( I_0 \) uniformly to \( f \) iff \( f(0) \) and \( f(1) \) are integers.)
2. Assume that $f \in AC(I_0)$. Show that $V(x) = V(f; [0, x])$ is also in $AC(I_0)$. 
3. Let $(X, M, \mu)$ be a measure space, and let $1 \leq p \leq \infty$. Let $\{f_n\} \subset L^p(\mu)$ with $\|f_n\|_{p'} \leq M < \infty$. Assume that $\{\int_X f_n \phi \, d\mu\}$ converges for every $\phi$ in a dense subset of $L^p(\mu)$. Show that $\{\int_X f_n \phi \, d\mu\}$ converges for every $\phi \in L^p(\mu)$. 
4. Let $f \in L^2(I_0)$, $\|f\|_2 = 1$ and $\int_0^1 f \, dm \geq \alpha > 0$. If $E_\beta = \{x \in I_0 : f(x) \geq \beta\}$ and $0 < \beta < \alpha$, then $|E_\beta| \geq (\alpha - \beta)^2$.

(Hint: $\alpha \leq \int_{E_\beta} + \int_{I_0 \setminus E_\beta}$.)
5. Let \((X, \mathcal{M}, \mu)\) be a measure space with \(\mu(X) < \infty\). Assume that \(||f_n||_p \leq M < \infty\), \(n = 1, 2, \ldots\), for some \(1 < p < \infty\), and that \(f_n \to f\) in measure, i.e., \(\mu\{x : |f(x) - f_n(x)| > \delta\} \to 0\) as \(n \to \infty\), for every \(\delta > 0\). Show that \(f_n \to f\) in \(L^1(\mu)\).

(Hint: \(\phi_n = |f - f_n|, E_{\delta,n} = \{x : \phi_n(x) > \delta\}\). Write \(\int_X \phi_n d\mu = \int_{X \setminus E_{\delta,n}} \phi_n d\mu + \int_{E_{\delta,n}} \phi_n d\mu\).)
6. Given \((X, \mathcal{M}, \mu)\), \(1 \leq p < \infty\), \(0 < \eta < p\). If \(f_n \to f(L^p)\) and \(g_n \to g(L^p)\), show that

\[
\lim_{n \to \infty} \int_X |f_n|^{p-\eta}|g_n|^{\eta} \, d\mu = \int_X |f|^{p-\eta}|g|^{\eta} \, d\mu.
\]