QUALIFYING EXAMINATION
MA 544

SPRING 1997

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $I_0 = [0, 1]$, and $L^p(\mathbb{R}), L^p(I_0)$ are the common L^p spaces over $\mathbb{R}, I_0$ with respext to Lebesgue measure dx. For a measurable subset A of $\mathbb{R}$, let $|A|$ denote the Lebesgue measure of A. All functions are assumed to be measurable.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let f be a nonnegative function in $L^1(I_0)$ such that for each $n = 1, 2, \cdots$

$$\int_0^1 f(x)^n dx = \int_0^1 f(x) dx.$$

Show that $f(x) = \chi_E(x)$ for some measurable set $E \subset I_0$.

2. Let $\alpha_n \in \mathbb{R}$ with $\sum |\alpha_n| < \infty$. If $\{r_n\}$ is an enumeration of the rationals in I_0 show that

$$\sum \frac{\alpha_n}{\sqrt{|x - r_n|}}$$

converges absolutely for a.e. $x \in I_0$.

3. Let $f_n : I_0 \to [0, \infty), n = 1, 2, \cdots$. Show that there are $\alpha_n > 0$ such that $\sum \alpha_n f_n(x)$ converges absolutely for a.e. $x \in I_0$.

4. Let $f \in L^2(\mathbb{R})$, and let $f_0(x) = x f(x)$. Show that

$$||f||_1 \leq \{8||f||_2||f_0||_2\}^{1/2}.$$

Hint: $\displaystyle\int_{\mathbb{R}} |f| = \int_{|x| \leq \alpha} |f| + \int_{|x| > \alpha} \frac{1}{|x|} |f_0(x)|$; apply Hölder's inequality and $\cdots$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ and let $E \subset \{x : f'(x) \text{ exists}\}$. If $|E| = 0$, show that $|f(E)| = 0$.

6. Let $\{r_k\}$ be a sequence of positive real numbers with $r_k \to 0$ and $\sum r_k = \infty$. Let $f : \mathbb{R} \to [0, \infty]$ and define inductively the sets A_k by

$$A_k = \{x \in \mathbb{R} : f(x) \geq r_k + \sum_{j<k} r_j \chi_{A_j}(x)\}.$$

Show that for every $x \in \mathbb{R}$,

$$f(x) = \sum_{k=1}^{\infty} r_k \chi_{A_k}(x).$$