QUALIFYING EXAMINATION
JANUARY 1998
MATH 544

Note: f is right continuous at x if $\lim_{\substack{h\to 0\\ h>0}} f(x+h) = f(x)$.

(16 pts) 1. a) Let μ be the set function on $[0,\infty)$ given by $\mu(\Lambda) = \sum_{\substack{n\in\Lambda\\ n\in\mathbb{N}}} \frac{(-1)^n}{n^2}$, where $\mathbb{N} = \{0,1,2,\ldots\}$, and sums are taken in the natural order on $\mathbb{N}$. Show that μ can be decomposed $\mu = \nu - \tau$, where ν and τ are both measures.

b. Let γ be the set function on $[0,\infty)$ given by $\gamma(\Lambda) = \sum_{\substack{n\in\Lambda\\ n\in\mathbb{N}}} \frac{(-1)^n}{n}$, with sums taken in the natural order on $\mathbb{N}$. Show that γ cannot be decomposed $\gamma = \nu - \tau$, where ν and τ are both measures.

(15 pts) 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$; suppose $x \to f(x,y)$ is Borel measurable for each $y \in \mathbb{R}$; and suppose that $y \to f(x,y)$ is right continuous for all $x \in \mathbb{R}$. Show that f itself is Borel measurable. [That is, if $\Lambda \in \mathcal{B}(\mathbb{R})$, then $f^{-1}(\Lambda) \in \mathcal{B}(\mathbb{R}^2)$, where $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^2)$ denote the Borel sets of $\mathbb{R}$ and $\mathbb{R}^2$ respectively.]

(16 pts) 3. Let X, Y, Z be topological spaces, and let $z = f(x,y)$ be a mapping from $X \times Y$ into Z. f is continuous in x if $x \to f(x,y)$ is a continuous mapping from X into Z, for each fixed $y \in Y$. f is continuous in y is defined analogously. f is jointly continuous in (x,y) if it's continuous as a mapping from $X \times Y$ into Z.

a) Show that if f is jointly continuous, then it is continuous in each variable separately.

b) Show that the converse to (a) is, in general, false. [Hint: consider $f(x,y) = \frac{xy}{x^2+y^2}$.]

(10 pts) 4. Let μ be a finite measure on $(\mathbb{R}, \mathcal{B})$.

a) Show that if f_n are measurable and $\lim f_n = f$ uniformly, then

$$\lim_{n\to\infty} \int f_n(x)\mu(dx) = \int f(x)\mu(dx).$$

b) Is the result in part (a) true if μ is not assumed to be finite? Justify your answer.

(16 pts) 5. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and assume $\int_0^T |f(x)|dx < \infty$. Let $F(t) = \int_0^t f(x)dx$ for $0 \le t \le T$.

a) Show that $\frac{d}{dt}F(t)$ exists a.e. (Lebesgue) on $[0, T]$, and is Borel measurable.

b) Let $G(t) = \int_\alpha^t f(x)dx$, for $0 < \alpha < T$, α fixed. Find the relation between $\frac{d}{dt}F(t)$ and $\frac{d}{dt}G(t)$ and justify your answer.

(15 points) 6. Let μ be a measure on $(\mathbb{R}, \mathcal{B})$, with $\mathcal{B}$ the Borel sets of $\mathbb{R}$. Let $\mu(\mathbb{R}) = 1$ and define an operator on measurable functions by

$$\pi(f) = \int_{\mathbb{R}} \min(|f(x)|, 1)\mu|dx).$$

a) Show that $f_n \to f$ in μ-measure if and only if $\pi(f_n - f) \to 0$.

b) If $(f_n)_{n\ge1}$ is a sequence of measurable functions such that $\sum_n \pi(f_n - f_{n+1}) < \infty$, show it converges in μ-measure.

c) Show that the sequence (f_n) of part (b) converges almost everywhere $(d\mu)$.

(12 points) 7. Consider the series $\sum_{n=1}^{\infty}(\sin(\pi n!e))^\alpha$ where $\alpha \in \mathbb{N}$. Determine if the series converges, and if it converges conditionally or absolutely, for each $\alpha \in \mathbb{N}$.