Each problem is worth 10 points.

1. Suppose that \( \{f_n\}_{n \geq 1} \) is pointwise bounded and equicontinuous on a compact set \( K \).
   (a) Show that \( \{f_n\}_{n \geq 1} \) is uniformly bounded on \( K \).
   (b) Show that \( f(x) = \inf \{f_n(x) : n \geq 1\} \) is uniformly continuous on \( K \).

2. Suppose that \( f \) is continuous on the interval \( 0 < x \leq 1 \).
   (a) Show that there exists a sequence of polynomials that converges pointwise to \( f \) on \( (0, 1] \).
   (b) State a necessary and sufficient condition on \( f(x) \) for \( x \in (0, 1] \) such that the convergence in (a) may be taken to be uniform on \( (0, 1] \). Show that the condition is necessary and sufficient.

3. If \( f_n \) is measurable for each \( n \geq 1 \), show that \( \limsup_{n \to \infty} f_n \) is measurable.

4. (a) Show that \( 0 \leq a \leq b \leq 2\pi \) implies \( \lim_{n \to \infty} \int_a^b \cos nt \, dt = 0 \).
   (b) Show that \( \lim_{n \to \infty} \int_0^{2\pi} f(t) \cos nt \, dt = 0 \) for every \( f \in L^1[0, 2\pi] \).

5. If \( f \in L^\infty[0, 1] \), show that \( \lim_{p \to \infty} \left( \int_0^1 |f|^p \, dx \right)^{1/p} = \|f\|_\infty \).

6. If \( f_n \to f \) a.e. on \( [0, 1] \) and, given any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( |E| < \delta \) implies \( \int_E |f_n| \, dx < \epsilon \), show that \( \lim_{n \to \infty} \int_0^1 |f - f_n| \, dx = 0 \).

7. (a) Let \( g_n(x) = \frac{e^{-x/n}}{n} \), \( x \geq 0 \), \( n \geq 1 \).
   (i) Show that \( g_n(x) \in L^1[0, \infty) \), \( n \geq 1 \).
   (ii) Show that the hypothesis of the Lebesgue Dominated Convergence Theorem is not satisfied for \( \{g_n\}_{n \geq 1} \).
   (b) Let \( h_n(x) = n \sin \left( \frac{x}{n} \right) \), \( 0 \leq x \leq \pi \), \( n \geq 1 \). Show that the hypothesis of the Lebesgue Dominated Convergence Theorem is satisfied for \( \{h_n\}_{n \geq 1} \).

8. Use the Vitali Covering Lemma to show that, if \( f \) is finite-valued and increasing on \( [0, 1] \), \( u > 0 \), \( \epsilon > 0 \), and \( m^* \) denotes Lebesgue outer measure, then

\[
m^* \left( \left\{ x \in [0, 1) : \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h} > u \right\} \right) \leq \frac{f(1) - f(0)}{u} + \epsilon.
\]