QUALIFYING EXAMINATION

AUGUST 2001

MATH 544 - Prof. Lempert

Each of the following seven problems is worth 5 points.

1. For a set $Y \subset \mathbb{R}$ and $r > 0$ let
$$B_r(Y) = \{x \in \mathbb{R} : \quad \exists y \in Y \quad \text{such that} \quad |x - y| < r\}.$$
Prove that $\bigcap_{r>0} B_r(Y) = \overline{Y}$.

2. Supposing the sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly, prove that the f_n's form a uniformly equicontinuous family.

3. Let $u : \mathbb{R} \to \mathbb{R}$ be arbitrary, and $v(x) = \inf_{|x-y|<1} u(y)$. Show that v is upper semicontinuous.

4. If $Z \subset \mathbb{R}$ is Lebesgue measurable, show that so is $W = \{2z : z \in Z\}$, and $m(W) = 2m(Z)$.

5. Suppose f_n, $n \in \mathbb{N}$, is a sequence of nonnegative measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$, $f_n \to f$ a.e., and there are subsequences f_{n_j}, f_{m_j} such that
$$\int_\Omega f_{n_j} \to 1, \quad \int_\Omega f_{m_j} \to 3, \qquad \text{as } j \to \infty.$$
State and justify the best upper and lower estimates for $\int_\Omega f$ based on the given information; show by examples that they cannot be improved.

6. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $f : \Omega \to \mathbb{R}$ measurable. Suppose there is a $c \in \mathbb{R}$ such that for all $X \subset \Omega$ of finite measure $|\int_X f| \le c$ holds. Prove that $f \in L^1(\Omega, \mathcal{A}, \mu)$.

7. Let $g : [a,b] \to \mathbb{R}$ be Lebesgue measurable, and suppose $\displaystyle\int_a^b g\psi = 0$ for all continuous $\psi : [a,b] \to \mathbb{R}$. Prove that $g = 0$ a.e.