QUALIFYING EXAMINATION
AUGUST 2003
MATH 544 - Prof. Davis

(15 pts) 1. Let \( f \) be an integer valued function on \( \mathbb{R} \). Show that \( \{ x : f \text{ is not continuous at } x \} \) is a Borel set.

(15 pts) 2. Let \( A \) and \( B \) be (not necessarily Lebesgue measurable) subsets of \( \mathbb{R} \) and let \( |e| \) stand for Lebesgue outer measure. Prove that if \( |A|_e = 1 \) and \( |B|_e = 1 \) and \( |A \cup B|_e = 2 \) then \( |A \cap B|_e = 0 \).

(15 pts) 3. Show, with proof, \( F'(1) < 0 \), where \( F(x) = \int_0^\infty \frac{e^{-xy}}{y^2 + 1} dy \).

(15 pts) 4. Prove that if \( f \) is a uniformly continuous function on \( \mathbb{R} \) and if \( h(x) = \int_{x-1}^{x+1} f(s) ds \) then \( h \) is uniformly continuous on \( \mathbb{R} \).

(20 pts) 5. Let \( g(x) \) be a continuous function on \([0, 1]\) satisfying \( g(0) = 0 \), \( g(1) \leq 1 \), and \( g(s) \leq g(t) \) if \( 0 \leq s < t \leq 1 \). Put \( \phi_n(x) = g(x)^n \). Prove that if \( f \) is a continuous function on \([0, 1]\), \( \lim_{n \to \infty} \int_0^1 f(x) d\phi_n(x) \) exists.

(20 pts) 6. Let \( f \) be a bounded Lebesgue measurable function on \( \mathbb{R} \). Put \( g(x) = \sup\{ a \in \mathbb{R} : |\{ y : y \in (x, x + 1) \text{ and } f(y) > a \}| > 0 \} \), where \( |\cdot| \) is Lebesgue measure (i.e. \( g(x) \) equals the essential supremum of \( f \) over \((x, x + 1)\)). Prove \( \liminf_{x \to 0} g(x) \geq g(0) \).