Qualifying Examination
January 2003
Math 544 — Prof. L. Brown

(30 pts) 1. You may use each part of this problem in the next part. Let (X, d) be a metric space.

a. For $\emptyset \neq F \subset X$, let $f(x) = d(x, F) = \inf\{d(x, y) \colon y \in F\}$. Show that f is continuous.

b. Let K and F be non–empty subsets of X such that K is compact. Show that there is p in K such that $d(p, F) = \inf\{d(x, y) \colon x \in K,\ y \in F\}$.

c. Assume $K \subset U \subset X$, where K is compact and U is open. Show that there is $r > 0$ such that $x \in K$ and $d(x, y) < r$ imply $y \in U$.

(30 pts) 2. a. Let $\{q_1, q_2, \ldots\}$ be an enumeration of the set of rational numbers q with $0 < q < 1$. Define $f\colon [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 2^{-n}, & x = q_n \\ 0, & \text{otherwise} \end{cases}$.
Show that f has bounded variation.

b. Give an example of a function $f\colon [0,1] \to \mathbb{R}$ such that $f = 0$ almost everywhere and f does not have bounded variation, and justify your answer.

(25 pts) 3. Assume that f_n is Lebesgue measurable for $n = 1, 2, \ldots,\ f_n \geq 0$, and $\sum_{n=1}^{\infty} \int f_n(x)dx < \infty$. Show that $f_n(x) \to 0$ for almost every x.

(30 pts) 4. In each case find $\lim_{n\to\infty} \int_0^\infty f_n(x)dx$ and justify your answer.

a. $f_n(x) = \begin{cases} \dfrac{\cos(\frac{x+1}{n})}{\sqrt{x}}, & 1 \le x \le n-1 \\ 0, & \text{otherwise} \end{cases}$

b. $f_n(x) = \begin{cases} \dfrac{\sin(\frac{x+1}{n})}{\sqrt{x}}, & n \le x \le 2n \\ 0, & \text{otherwise} \end{cases}$.

c. $f_n(x) = \begin{cases} \dfrac{\sin(1+\frac{x}{n})}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(30 pts) 5. Assume $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $\displaystyle\int |f(x)|^p dx < \infty$, and $\int |g(x)|^q dx < \infty$.

a. For x in $\mathbb{R}$ let $K_x(y) = f(x-y)g(y)$. Show that K_x is Lebesgue integrable.

b. Let $h(x) = \int f(x-y)g(y)dy$. Show that h is bounded.

c. Show that h is continuous.

(25 pts) 6. Assume that f_n is absolutely continuous on $[0,1]$ for $n = 1, 2, \ldots$, there is a function g in $L^1([0,1])$ such that $\|f_n' - g\|_1 \to 0$, and that the sequence (f_n) is Cauchy in $L^1([0,1])$. Show that there is an absolutely continuous function h on $[0,1]$ such that (f_n) converges uniformly to h.