QUALIFYING EXAMINATION

AUGUST 2004

MATH 544 - Prof. Dadarlat

There are six questions. Use the space provided for solutions.

1. Prove that

$$m^*(E_1) + m^*(E_2) \leq 2\, m^*(E_1 \triangle E_2) + 2\, m^*(E_1 \cap E_2)$$

where m^* is the Lebesgue outer measure on $\mathbb{R}$ and $E_1, E_2 \subset \mathbb{R}$.

(10 pts)

2. Show that the following limit exists

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+1/n)-\cos x}{x^{3/2}}\,dx.$$

(25 pts)

3. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Show that if $0 \leq b \leq m(A)$, then there is a Lebesgue measurable set $B \subset A$ with $m(B) = b$.

(15 pts)

4. A Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$\int_E f(x)\,dx = 0$$

for all Lebesgue measurable sets $E \subset \mathbb{R}$ with $m(E) = \pi$. Prove or disprove that $f = 0$ a.e.

(20 pts)

5. Find all the functions $f : [0,1] \to \mathbb{R}$ with bounded variation satisfying

$$f(x) + (T_0^x f)^{1/2} = 1, \quad \forall x \in [0,1],$$

and

$$\int_0^1 f(x)\,dx = 1/3.$$

(Hint: Prove first that f is monotonic.)
($T_a^b f$ *stands for the total variation of* f *on the interval* $[a,b]$.)

(15 pts)

6. Show that the sets

$$S_1 = \{f \in L^2[0,1] : \int_0^1 (1 - x^2)\, f(x)\, dx > 0\} \qquad (10pts)$$

and

$$S_2 = \{f \in L^3[0,1] : \int_0^1 (1 - 2x^3) f(\sin x)\, dx > 0\} \qquad (10pts)$$

are open in $L^2[0,1]$ and respectively $L^3[0,1]$.