QUALIFYING EXAMINATION
JANUARY 2006
MATH 544 - Prof. Sa Barreto

INSTRUCTIONS **(READ THEM CAREFULLY)**

1) This exam has nine questions worth a total 200 points. The number in parenthesis at the beginning of a question, or an item in a question, indicates the value of that particular problem.
2) Answer one problem per page, and clearly indicate which problem you are solving. If you use more than one page in a problem, clearly indicate what you are doing. If you choose to ignore this request, I will not be responsible for finding your solution, or part of it, in another sheet.
3) Justify your solutions. Answers without justifications will have ZERO value.

1) (10 points) Let $f : [0,1] \longrightarrow [0,1]$. Prove that f is Lebesgue integrable if and only if there exists a monotone sequence of measurable simple functions bounded from above by f and converging to f in $[0,1]$.

2) a (20 points) Give an example of a sequence $f_n : [0,1] \longrightarrow [0,1]$ of bounded Riemann integrable functions such that

$$\lim_{m,n\to\infty} \int_0^1 |f_n - f_m|\, dx = 0,$$
$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in [0,1]$$

with f bounded but not Riemann integrable.

b) (10 points) Is f necessarily Lebesgue integrable?

3) a (20 points) Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set with positive and finite measure. Let χ_A be the characteristic function of A. Let

$$\phi(x) = \int_{\mathbb{R}^n} \chi_A(y)\chi_A(x+y)\, dy.$$

Prove that ϕ is continuous.

b)(10 points) Use a) to show that the set

$$A - A = \{x \in \mathbb{R}^n : x = y_1 - y_2, \quad y_1, y_2 \in A\}$$

contains a neighborhood of the origin.

4)(20 points) Prove or give a counter-example to the following: Let $f_n \in L^1([0,1])$, $n = 1, 2, ...,$ and suppose that $f_n \longrightarrow 0$ in $L^1([0,1])$. Then $f_n \longrightarrow 0$ a.e.

5) (20 points) Show that if $f \in L^p([0,1]) \cap L^r([0,1])$, with $p < r$, then $f \in L^s([0,1])$ for all $s \in [p,r]$.

6) a (10 points) Show that

$$L^\infty((0,1)) \subset \bigcap_{p\geq 1} L^p((0,1)).$$

b) (10 points) Is equality true?

7) (20 points) Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Compute the limit

$$\lim_{h\to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx..$$

8) Let

$$f(x) = \begin{cases} x^p \sin\left(\frac{1}{x^q}\right) & \text{if} \quad x \in (0,1] \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

a) (20 points) Show that if $p > q > 0$ then f is absolutely continuous.

b) (20 points) However, if $0 < p \leq q$ show that f is not of bounded variation.

9) (10 points) Let $f_n : [0,1] \longrightarrow [-1,1]$, $n = 1, 2, ...$, be a sequence of absolutely continuous functions. Suppose that $f_n \longrightarrow f$ uniformly. Is f absolutely continuous?