Math 544
Qualifying Examination
August, 2008 – Prof. Davis

(15) 1. Show that if $E \subset \mathbb{R}$ and if $| \, |_e$ stands for outer measure then

$$|E|_e = \sum_{n=-\infty}^{\infty} |E \cap [n, n+1]|_e.$$

Also prove that if $O_i,\ i \geq 1$, are open subsets of $\mathbb{R}$ satisfying $\bigcup_{i=1}^{\infty} O_i = \mathbb{R}$ then

$$\lim_{k\to\infty} \left| \left(\bigcup_{i=1}^{k} O_i \right) \cap E \right|_e = |E|_e.$$

(15) 2. Let $f_1, f_2, \ldots$ be functions on $\mathbb{R}^n$ such that $\displaystyle\int_{\mathbb{R}^n} f_k = 1$, $k \geq 1$, and $0 \leq f_k \leq \dfrac{1}{k}$.

Prove $\displaystyle\int_{\mathbb{R}^n} \sup_{k\geq 1} f_k = \infty$.

(Here integrals are with respect to n dimensional Lebesgue measure.)

(15) 3. Let $c(x)$ be the Cantor function on $[0, 1]$.
(So $c(x)$ is a continuous nondecreasing function which equals $\frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, equals $\frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, equals $\frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$, etc.)

Clearly $c'(x) = 0$ for $x \in (0, 1), x \notin C$, where C, the Cantor set, is the complement of the union of $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$, and all the rest of the middle thirds. Are there any $x \in (0, 1) \cap C$ such that $c'(x)$ exists (and is finite)? Prove your answer.

(15) 4. Let (S, Q, m) be a measure space satisfying $m(S) < \infty$. Let f be a measurable function on S satisfying $|f| < 1$. Prove that either $\lim_{n\to\infty} \int_S (1 + f + \cdots + f^n) dm$ exists or that this limit is $+\infty$.

5.

(5) i) Prove that if f is a continuous function on $[0,1]$ such that $f'(x) = 0$ for all but perhaps a finite number of x then f is a constant function, i.e. $f(x) = f(y)$, $x, y \in [0,1]$.

(15) ii) Prove that if g is a continuous function on $[0,1]$ satisfying $g(s) \leq g(t)$ if $s \leq t$ such that $g'(x) = 0$ for all but perhaps a countable number of x then g is a constant function.

(20) 6. Let f be a function on $(-\infty, \infty)$ such that given $\varepsilon > 0$ there is a polynomial $p(x)$ such that $|p(x) - f(x)| < \varepsilon$, $x \in \mathbb{R}$. Prove f is a polynomial.