Problem 1. (20 points) Let \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) be a function such that for some \( s > n/2 \) its Fourier transform \( \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} f(x) dx \) is such that
\[
\int_{\mathbb{R}^n} (1 + |\xi|^2) s |\hat{f}(\xi)|^2 d\xi < \infty .
\]
1) Prove that \( \hat{f} \in L^1(\mathbb{R}^n) \).
2) You can assume as known that if \( f, \hat{f} \in L^1(\mathbb{R}^n) \), then the following inversion formula holds
\[
f(x) = \int_{\mathbb{R}^n} e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi) d\xi , \quad \text{for a.e. } x \in \mathbb{R}^n .
\]
Use (2) and Part 1) to prove that if \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) is a function such that (1) holds for some \( s > n/2 \), then \( f \) coincides a.e. with a continuous function on \( \mathbb{R}^n \).
**Problem 2.** (20 points) Prove that if \( f(x) = \chi_R(x) \), where \( R = (a, b) \times (c, d) \subset \mathbb{R}^2 \), and \( \chi_R \) denotes the characteristic function of \( R \), then the Fourier transform of \( f \) verifies \( \hat{f}(\xi) \to 0 \) as \( |\xi| \to \infty \).

**Hint:** Start by computing the Fourier transform \( \hat{\chi}_{(-A,A)} \) of a symmetric interval on the line, then...
Problem 3. (25 points) Consider the function $K(x) = (4\pi)^{-n/2} \exp(-(|x|^2)/4)$, $x \in \mathbb{R}^n$, and let $K_t(x) = t^{-n/2}K(x/\sqrt{t})$, $t > 0$.

Given a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define

$$P_t f(x) = K_t * f(x) = \int_{\mathbb{R}^n} K_t(x - y)f(y)dy.$$ 

1) Prove that for every $1 \leq p < \infty$, one has

$$\lim_{t \to 0^+} \| P_t f - f \|_{L^p(\mathbb{R}^n)} = 0,$$

for every $f \in L^p(\mathbb{R}^n)$, and that moreover for every $1 \leq p \leq \infty$

$$\| P_t f \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)}.$$

2) Prove that if $1 \leq p < \infty$ one also has $P_t : L^p(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ with

$$\| P_t f \|_{L^\infty(\mathbb{R}^n)} \leq (p')^{-n/2p'} (4\pi t)^{-n/2p} \| f \|_{L^p(\mathbb{R}^n)},$$

where $1/p + 1/p' = 1$. Here, one must take $(p')^{-n/2p'} = 1$ when $p = 1$. 
Problem 4. (20 points) Let

\[ N(x, y, z) = \left((x^2 + y^2)^2 + z^2\right)^{1/4}, \quad (x, y, z) \in \mathbb{R}^3. \]

Using the change of variable

\[ x = \rho \cos \theta \sin^{1/2} \phi, \quad y = \rho \sin \theta \sin^{1/2} \phi, \quad z = \rho^2 \cos \phi, \]

with \((\rho, \theta, \phi) \in \Omega = (0, \infty) \times (0, 2\pi) \times (0, \pi),\) identify all values of \(p \in \mathbb{R}\) for which \(N^{-p} \in L^1(E),\)

where \(E = \{(x, y, z) \in \mathbb{R}^3 \mid N(x, y, z) < R\},\) with \(R > 0\) fixed.
Problem 5. (20 points) Use Lebesgue dominated convergence theorem to compute the limit

\[
\lim_{k \to \infty} \int_{B(0,1)} \frac{1 - e^{-|x|^2}}{|x|^{n+1}} \, dx,
\]

where \( B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \).
Problem 6. (25 points) Let $S^{n-1} = \{ \omega \in \mathbb{R}^n \mid |\omega| = 1 \}$ be the unit sphere in $\mathbb{R}^n$ and for every $\lambda > 0$ consider the function

$$f(x) \overset{def}{=} \int_{S^{n-1}} e^{-i\sqrt{\lambda}<x,\omega>} \, d\sigma(\omega), \quad x \in \mathbb{R}^n,$$

where $d\sigma(\omega)$ indicates the induced Lebesgue measure on $S^{n-1}$.

1) Prove that $f \in C^2(\mathbb{R}^n)$ and satisfies the equation

$$\Delta f = -\lambda f,$$

where $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$.

2) Prove that if $n = 3$, then

$$f(x) = 4\pi \frac{\sin(\sqrt{\lambda}|x|)}{\sqrt{x}|x|}.$$ 

Hint: Use spherical coordinates in $\mathbb{R}^3$ and then express the integral on the unit sphere $S^2 = \{ \omega \in \mathbb{R}^3 \mid |\omega| = 1 \}$ as an iterated integral over the one-parameter family of circles $L_\theta = \{ \omega \in S^2 \mid <\omega, \frac{x}{|x|}> = \cos \theta \}$ forming an angle $\theta$ with the fixed direction $\frac{x}{|x|} \in S^2$. 
