Each problem is worth 5 points.

1. Let \( a < b \) be real numbers, \( g_i : \mathbb{R} \to [a, b] \) arbitrary functions, \( i \in \mathbb{N} \), and \( h : \mathbb{R} \to \mathbb{R} \) continuous. Supposing that the \( g_i \) converge uniformly, prove that \( h \circ g_i \) also converge uniformly.

Does the same hold if instead of \( g_i : \mathbb{R} \to [a, b] \) we require only \( g_i : \mathbb{R} \to \mathbb{R} \)?

2. Suppose \( f \in L^1(\Omega, \mathcal{A}, \mu) \) and \( f(x) \neq 0 \) for almost every \( x \in \Omega \). Prove that \( \mu \) is \( \sigma \)-finite.

3. Let \( f \) be an everywhere finite measurable function on a measure space \((\Omega, \mathcal{A}, \mu)\), such that for every continuous function \( \alpha : \mathbb{R} \to \mathbb{R} \) the composition \( \alpha \circ f \) is integrable. Prove that \( \operatorname{ess sup} |f| < \infty \).

4. Let \( a < b \) be real numbers and \( \phi_n : [a, b] \to \mathbb{R} \) a sequence of increasing, absolutely continuous functions. Show that if the series \( \sum_{n=1}^{\infty} \phi_n \) converges pointwise, then its sum is also absolutely continuous.

5. Fix \( p \in [1, \infty) \), and for \( j = 1, 2, \ldots \) define functions \( \omega_j : l^p \to \mathbb{R} \) by

\[
\omega_j(x_1, x_2, \ldots) = \sum_{i=j}^{\infty} |x_i|^p.
\]

Let \( C \subset l^p \) be a closed, bounded set, such that the functions \( \omega_j|C \) converge uniformly. Prove that any sequence in \( C \) contains a convergent subsequence.

6. Let \((\Omega, \mathcal{A}, \mu)\) be a finite measure space, \( p_0 \in [1, \infty) \), and \( \psi \in L^{p_0}(\Omega, \mathcal{A}, \mu) \). Prove that the function

\[
[1, p_0] \ni p \mapsto \|\psi\|_p
\]

is continuous.

7. Suppose \( X_1 \subset X_2 \subset \ldots \subset \mathbb{R} \) is an increasing sequence of subsets and \( X = \bigcup_{k=1}^{\infty} X_k \).

Denoting outer Lebesgue measure by \( m^* \), prove

\[
\lim_{k \to \infty} m^*(X_k) = m^*(X).
\]