Qualifying examination MA544, Summer 2012

Name: ____________________

1. (30 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f and g be real-valued integrable functions on X with
$$\int_X f d\mu = \int_X g d\mu$$
Show that either
(a) $f = g$ a.e. on X, or
(b) there exists a set $E \in \mathcal{M}$ such that $\int_E f d\mu > \int_E g d\mu$.

2. (25 points) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Show that, for every $\varepsilon > 0$, there exists a polynomial p and $a, b \in \mathbb{R}$ such that $\int_{\mathbb{R}} |f - p\chi_{[a,b]}| d\lambda < \varepsilon$.

3. (a) (20 points) Let (X, d) be a compact metric space and $f : X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ holds for all $x, y \in X$. Then show that f is onto.

   (b) (10 points) Does the conclusion remain true if X is not assumed to be compact?

4. (a) (15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be an integrable function. Show that for each $\varepsilon > 0$ there exists some $\delta > 0$ (depending on ε) such that $|\int_E f d\mu| < \varepsilon$ holds for all measurable sets with $\mu(E) < \delta$.

   (b) (15 points) Let $\{A_n\}$ be a sequence of measurable sets satisfying $0 < \mu(A_n) < \infty$ for each n and $\lim_{n\to\infty} \mu(A_n) = 0$. Fix $1 < p < \infty$ and let $g_n = (\mu(A_n))^{-\frac{1}{q}} \chi_{A_n}, n = 1, 2, ...,$ where $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\lim_{n\to\infty} \int_X f g_n d\mu = 0$ for each $f \in L^p(X, \mathcal{M}, \mu)$.

5. (25 points) Consider the measure space $(X, \mathcal{M}, \mu)$. Let g be an integrable function and let $\{f_n\}$ be a sequence of integrable functions such that $|f_n| \leq g$ a.e. holds for all n. Show that if f_n converges to f in measure then f is an integrable function and $\lim_{n\to\infty} \int_X |f_n - f| d\mu$=0

6. (30 points) Consider the function $f(x) = \frac{\sin^2 x}{x^2}$ on $[0, \infty)$.

   (a) Show that f is Lebesgue integrable.

   (b) Compute the Lebesgue integral $\int_0^\infty \frac{\sin^2 x}{x^2} d\lambda = \frac{\pi}{2}$.

   (Hint: You can use $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, in the sense of an improper Riemann integral. Recall that the Lebesgue integral $\int_0^\infty \frac{\sin x}{x} d\lambda$ does not exist.)

7. (30 points) Show that the function $f : [0, 1] \to \mathbb{R}$ is Lipschitz with constant M (i.e. $|f(x) - f(y)| \leq M|x - y| \ \forall x, y$) if and only if there is a sequence of continuously differentiable functions $f_n : [0, 1] \to \mathbb{R}$ such that

(i) $|f_n'(x)| \leq M$, $\forall x \in [0,1]$

(ii) $f_n \to f$ pointwise on $[0,1]$

(Hint: In one direction, you could start by using Lusin's Theorem to construct a sequence of continuous functions $g_n : [0,1] \to \mathbb{R}$ such that $g_n \to f'$ a.e. and $|g_n(x)| \leq M, x \in [0,1], n = 1,2,3,...$).