QUALIFYING EXAMINATION
August 2013
MATH 544–R. Bañuelos

Instructions: There are a total of 6 problems. A problem appears on each of the following pages. Problems are worth **20 points** each. Use the space provided for the solutions, using back pages as needed.

Problem 1.

(i) **(5-pts)** Prove that any function f of bounded variation on $[0,1]$ is Riemann integrable. (You may appeal to the characterization of Riemann integral functions in terms of Legesgue measure!)

(ii) **(10-pts)** Let m denote the Lebesgue measure on $\mathbb{R}$. Let $A \subset \mathbb{R}$ be Lebesgue measurable. A point $x \in \mathbb{R}$ is called a density point of A if

$$\lim_{\varepsilon \to 0} \frac{m\left(A \cap [x, x+\varepsilon]\right)}{|\varepsilon|} = 1$$

where m stands for the Lebesgue measure. Prove that almost all points of the set A are density points.

(iii) **(5-pts)** Find a sequence $\{f_n\}$ of Borel measurable functions on $\mathbb{R}$ which decreases uniformly to 0 but such that for all n,

$$\int_{\mathbb{R}} f_n dx = \infty.$$

(Here, as usual $dx = dm$.)

Problem 2. **(20-pts)** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $1 < p < \infty$. Suppose f_n is a sequence of measurable functions in $L^p(\mu)$ with $\|f_n\|_p \leq 1$ for all n and $f_n \to f$ a.e. Prove that

$$\int_X f_n g d\mu \to \int_X f g d\mu$$

for all $g \in L^q(\mu)$ where q is the conjugate exponent of p.

Problem 3. (20–pts) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $g_n : X \to \mathbb{R}$ be a sequence of measurable functions satisfying:

$$(i) \qquad \int_X |g_k|^2 d\,\mu \leq 100, \quad \text{for all k}$$

and

$$(ii) \qquad \int_X g_j g_k d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that

$$\lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} g_k = 0, \quad a.e.$$

for all $\beta > 3/2$.

Problem 4. **(20–pts)** Let f be Lebesgue measurable on $[0, 1]$ with $f > 0$ a.e. Suppose $\{E_k\}$ is a sequence of measurable sets in $[0, 1]$ with the property that $\int_{E_k} f(x)dx \to 0$, as $k \to \infty$. Prove that $m(E_k) \to 0$, as $k \to \infty$.

Problem 5. (10 pts each) Compute the following limits, fully justifying all your steps.

1.
$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin\left(\frac{x}{n}\right)}{x\left(1+x^2\right)} dx$$

2.
$$\lim_{n\to\infty} \int_0^\infty \sin\left(\frac{x}{n}\right)\left(1+\frac{x}{n}\right)^{-n} dx$$

Problem 6. **(20–pts)** Let $f \in L^2[0,1]$ be such that

$$\int_0^1 f(x)g(x)dx = 0$$

for all continuous functions g with the property that

$$\int_0^1 g(x)dx = \int_0^1 xg(x)dx = 0.$$

Prove that there is a linear function $l(x) = a + bx$ such that $f(x) = l(x)$, for almost all $x \in [0,1]$.