Real Analysis Qualifying Exam, January 2014

Name:
Student Number:

1. (20 pts.) Given $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $p, q > 1$ and $1/p + 1/q = 1$, consider the convolution

$$f * g(y) = \int_{\mathbb{R}} f(x-y)g(y)\, dy\ .$$

Prove that $f * g$ is well-defined, continuous, and bounded. Also prove that $\lim_{|x|\to\infty} f * g(x) = 0$.

2. (20 pts.) Let ϕ be a bounded linear functional on $L^2(\mathbb{R})$. Prove directly from the definition that $F(x) = \phi(\chi_{[0,x]})$ is absolutely continuous, where $\chi_{[0,x]}$ is the characteristic function of $[0, x]$. Use the Riesz Representation Theorem to find a formula for the derivative of $F(x)$ almost everywhere.

3. (20 pts.) Suppose that $1 < p < \infty$. We say that a sequence (f_n) in $L^p([0,1])$ converges weakly to $f \in L^p([0,1])$ if $\phi(f_n) \to \phi(f)$ for every bounded linear functional ϕ on $L^p([0,1])$. Assume that $\|f_n\| \leq 1$ and that $f_n \to 0$ almost everywhere. Prove that f_n converges weakly to 0. (Hint: use Egorov's Theorem.)

4. (20 pts.) Suppose that $A, B \subseteq [0, 1]$ are measurable sets each of Lebesgue measure at least $1/2$. Prove that there exists an $x \in [-1, 1]$ such that the measure of $(A + x) \cap B$ is at least $1/10$.

5. (20 pts.) Suppose that $p > 4/3$ and that $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{t \to 0^+} \int_0^t x^{-1/4} f(x)\, dx = 0 \ .$$

6. (20 pts.) Suppose that X is a normed vector space. Show that X is complete if and only if every absolutely convergent series converges in norm. ($\sum x_n$ is absolutely convergent if $\sum \|x_n\| < \infty$.)