FALL 2017

Qualifying Exam - MA 553

In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work.

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1. (a) (10 points) How many 2-subgroups are in A_4.

   (b) (10 points) Show that any Sylow 2- subgroup in S_5 is isomorphic to D_8.

2. Assume G contains a normal Sylow 2-subgroup P which is a cyclic group, and such that G/P is cyclic.

   (a) Show that the action of G by conjugation on P its trivial.(10 points) (Hint: consider the induced action of G/P on P.)

   (b) Show that G is abelian. (10 points)

3. (20 points) Show that any group G of order $32 \cdot 31$ is solvable.

4. (1) Find all (up to isomorphism) abelian groups of order 40 (10 points).
   (2) Find the number of elements of order 2 in each of them (10 points).

5. (15 points) Let R be an integral domain and F its field of fractions. Let P be a prime ideal in R and $R_P = \{\frac{a}{b} \mid a, b \in R, b \notin P\} \subset F$. Show that R_P has a unique maximal deal.

6. (20 points) Let $(R, +,)$ be a commutative ring with $1 \neq 0$ containing a unique maximal ideal M (i.e. R is a local ring). Show that the following conditions are equivalent:

   (a) $a \in M$

   (b) $1 + ca$ is invertible for any $c \in R$.

7. (25 points) Prove that

(a) $\mathbb{Q}[x]/(x-a) \simeq \mathbb{Q}$ for any $a \in \mathbb{Q}$. (5 points)

(b) $\mathbb{Q}[x]/(x^2+1) \simeq \mathbb{Q}[i]$ (5 points)

(c) Find a simpler form of $\mathbb{Q}[x]/(x^6-x^3)$ (5 points).
Prove that

(d) $\mathbb{Z} \to \mathbb{Z}[i]/(4+i), \quad a \mapsto a+(4+i)$ is surjective.(5 points)

(e) $\mathbb{Z}[i]/(4+i) \simeq \mathbb{Z}_{17}$ (5 points).

8. (35 points)

Let L_1 be the splitting field $\mathbb{Q}$ of $x^3 - 2$ over $\mathbb{Q}$, and L_2 be the splitting field $\mathbb{Q}$ of $x^2 + 1$ over $\mathbb{Q}$.

(a) Deteremine $[L_1 : \mathbb{Q}]$ (5 points)

(b) Find the Galois groups $Gal(L_1/\mathbb{Q})$, $Gal(L_2/\mathbb{Q})$ (5 points).

(c) Find all subfields $K_1 \subset L_1$ such that $[K_1 : \mathbb{Q}] = 2$ (5 points)

(d) Show that $L_1 \cap L_2 = \mathbb{Q}$. (5 points)

(e) Determine the degree $[L : \mathbb{Q}]$, and the Galois group of the splitting field L over $\mathbb{Q}$ of $f(x) = (x^3 - 2)(x^2 + 1)$ (5 points).

(f) Determine all the subgroups of index 3 in $Gal(L_2/\mathbb{Q})$. (5 points)

(g) Determine all the subfields $K \subset L$, such that $[K : \mathbb{Q}] = 4$ (5 points).

9. (25 points) Let K be the splitting field of

$$f(x) = (x^3 - 1)(x^4 - 1)(x^3 + x^2 + 1)(x^3 + x + 1)$$

over F_2.

(a) Decompose $f(x)$ into irreducibles. (5 points)

(b) Find $[K : F_2]$ (5 points)

(c) Find the Galois group $Gal(K/F_2)$ and its generator(s) (5 points).

(d) Find all the subgroups of $Gal(K/F_2)$. (5 points)

(e) Find all the intermediate fields $F_2 \subseteq E \subseteq K$ (5 points).