Qualifying Examination

MA 553

Time: 2 hours

August 11, 2015

Instructor: F. Shahidi

Name & PUID ______________________________

1	
2	
3	
4	
5	
6	
7	
Total	

Name & PUID ______________________________

(20 pts) 1. Prove that every group of order 187 is cyclic.

Name & PUID ________________________________

(25 pts) 2. Let p be a prime number, $p \geq 7$, and fix a positive integer m. Show that every group of order $6p^m$ is solvable.

Name & PUID ________________________________

(5 pts) 3. a) Give the definition of a euclidean domain.

(25 pts) b) Show that $\mathbb{Z}[\sqrt{-2}]$ is a euclidean domain. Is $\mathbb{Z}[\sqrt{-2}]$ a principal ideal domain (PID)? Prove this. Quoting a theorem is not acceptable.

Name & PUID ______________________________

Name & PUID ______________________________

4.

(10 pts) a) Let R be a unique factorization domain and consider

$$f(x,y) = x^7 + yx^5 + yx^3 + 3yx + y \in R[x,y].$$

Show that $f(x,y)$ is irreducible in $R[x,y]$.

(10 pts) b) Let $K = F(x^7/x^5 + x^3 + 3x + 1)$, where F is a field. Determine $[F(x)\colon K]$.

Name & PUID ______________________________

Name & PUID ______________________________

(30 pts) 5. Let K be the splitting field of $g(x) = x^p - x - a$, over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, and $a \in \mathbb{F}_p$, $a \neq 0$. Show that the Galois group $G(K/\mathbb{F}_p)$ is a cyclic group of order p, generated by the automorphic sending α to $\alpha + 1$, for every root α of $g(x)$.

Name & PUID ______________________________

Name & PUID ______________________________

6. Let $f(x) = x^3 - 2$, $g(x) = x^2 - 2$ and $h(x) = f(x)g(x)$ as polynomials over $\mathbb{Q}$.

(10 pts) a) Determine $\mathrm{Gal}_{\mathbb{Q}}(f)$, Galois group of a splitting field of f.

(20 pts) b) Determine all the fields L with $\mathbb{Q} \subset L \subset \mathrm{Gal}_{\mathbb{Q}}(f)$.

(5 pts) c) Determine $\mathrm{Gal}_{\mathbb{Q}}(g)$.

(15 pts) d) Use a), b) and c) to determine $\mathrm{Gal}_{\mathbb{Q}}(h)$.

Name & PUID ______________________________

Name & PUID ______________________________

Name & PUID ______________________________

7.

(5 pts) a) Define an extension by radicals.

(5 pts) b) Verify that $K = \mathbb{Q}(\sqrt[3]{1+\sqrt{7}})$ is an extension by radicals.

(10 pts) c) Let $\alpha = \sqrt[3]{1+\sqrt{7}}$ and $\beta = \sqrt[3]{1-\sqrt{7}}$. Show that $F = \mathbb{Q}(\alpha, \beta, \omega)$ is a Galois closure of $K/\mathbb{Q}$, where $\omega^2+\omega+1 = 0$. Moreover show that $F/\mathbb{Q}$ is an extension by radicals.

(5 pts) d) Is $F/\mathbb{Q}$ solvable? Justify your answer.

Name & PUID ______________________________