1. (14 points) Determine all groups of order 255 up to isomorphism. (Hint: Show that the center is not trivial.)

2. (13 points) Let $G$ be a group of order 375 that has a cyclic 5-Sylow subgroup. Show that $G$ is cyclic.

3. (16 points) We write $\mathbb{Z}[i] \subset \mathbb{C}$ for the ring of Gaussian integers and $\mathbb{Z}[X]$ for the polynomial ring in one variable.
   (a) Show that there is an isomorphism of rings $\mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2 + 1)$.
   (b) Determine the number of distinct ideals of the ring $\mathbb{Z}[i]$ that contain $3 + i$.

4. (14 points) Let $k$ be a field, $f$ an irreducible polynomial in $k[X]$, and $k \subset K$ a normal field extension. Consider a factorization $f = f_1 \cdots f_s$, where $f_i$ are irreducible polynomials in $K[X]$. Show that all $f_i$ have the same degree.

5. (17 points) Let $K$ be a field of characteristic zero with algebraic closure $\overline{K}$, and let $\alpha$ be an element of $\overline{K}$. Let $L$ be a maximal intermediate field of $K \subset \overline{K}$ not containing $\alpha$, i.e., a subfield of $\overline{K}$ with $K \subset L$ and $\alpha \notin L$ so that $L = L'$ for every subfield $L'$ of $\overline{K}$ with $L \subset L'$ and $\alpha \notin L'$. Show that the field extension $L \subset \overline{K}$ is Galois and Abelian. (Hint: First show that every finite Galois extension of $L$ in $\overline{K}$ is cyclic.)

6. (14 points) Let $k \subset K$ be a finite Galois extension and $L$ an intermediate field, $k \subset L \subset K$, so that $[K : k]$ does not divide $[L : k]$! Show that there exists an intermediate field $L'$, $L \subset L' \subset K$, so that $L' \neq K$ and $k \subset L'$ is Galois.

7. (12 points) Let $p$ be a prime number and let $K \subset \mathbb{C}$ be the splitting field of $X^p - 6$ over $\mathbb{Q}$.
   (a) Determine the field $K$ and the degree $[K : \mathbb{Q}]$.
   (b) Determine all subfields $L$ of $K$ with $[L : \mathbb{Q}] = p - 1$. 