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Instructions: Give a complete solution to each problem. Be sure to show all your work. You may cite any result except the one you are asked to prove. If a result has a name, you may refer to it by name. Otherwise, be sure to indicate the content of the result. The exam is graded 0-200 points.

1. **(20 points)** Let G be a group and suppose H is a subgroup such that $x^2 \in H$ for all $x \in G$. Prove H is a normal subgroup of G.

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2. **(9 points each)** Denote by S_n the symmetric group on n letters and let A_n be the subgroup of even permutations in S_n. Recall A_n is a simple group if $n \geq 5$. Also, you may take as a fact that if $H \subset S_n$ is simple and $|H| > 2$, then $H \subset A_n$.

(a) Show any homomorphism $\varphi : A_6 \to S_4$ is trivial.

(b) Show there is no subgroup G of A_6 with $[A_6 : G] = 4$.

(c) Suppose G is a group of order 90 with no normal subgroup of order 5. Show there is a non-trivial homomorphism $\varphi : G \to S_6$.

(d) Show there is no simple group of order 90.

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3. **(20 points)** Let R be a commutative ring with identity, and suppose G is a finite subgroup of $R^{\times}$, the group of units. Show that if R is an integral domain, then G is cyclic.

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4. **(12 points each)** Let R be an integral domain. Suppose $I_1, I_2, \ldots, I_m$ are ideals in R.

   (a) Suppose $I_1 \cap I_2 \cap \cdots \cap I_m = 0$. Show there is some $1 \leq j \leq m$ with $I_j = 0$.

   (b) Show the conclusion of (a) may be false for an intersection of infinitely many ideals.

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5. **(20 points)** Let F be a field . Prove the polynomial ring $F[x]$ has infinitely many maximal ideals.

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6. **(15 points each)** Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree n and let $K/\mathbb{Q}$ be a splitting field of $f(x)$. Suppose the Galois group $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_n$, the symmetric group on n letters.

   (a) Prove $f(x)$ is irreducible over $\mathbb{Q}$.

   (b) Let $\alpha \in K$ be a root of $f(x)$. Suppose σ is automorphism of the field $\mathbb{Q}(\alpha)$. Prove $\sigma = 1$.

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7. **(14 points)** Prove there is a Galois extension $K/\mathbb{Q}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/7\mathbb{Z}$.

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8. **(9 points each)** Let F be a field with $\mathbb{C} \supset F \supset \mathbb{Q}$, and $F/\mathbb{Q}$ an abelian (Galois) extension. Let $\alpha \in F$, with minimal polynomial $f(x) \in \mathbb{Q}[x]$. Let $\tau : \mathbb{C} \to \mathbb{C}$ be complex conjugation. Assume $|\alpha| = 1$ ($\mathbb{C}$–absolute value). Recall an **algebraic integer** is an element of $\mathbb{C}$, which is a root of a monic polynomial in $\mathbb{Z}[x]$.

   (a) Prove $\tau \in \mathrm{Gal}(F/\mathbb{Q})$.

   (b) Show $|\beta| = 1$ for every root β of $f(x)$.

   (c) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Show $|a_i| \le 2^n$ for $i = 0, 1, \ldots, n-1$. (Hint: You may use $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.)

   (d) Show F has only finitely many algebraic integers of absolute value 1.