1. Let $\mathbb{F}_p$ denote the prime field with $p$ elements.

(i) How many monic polynomials are there in $\mathbb{F}_p[x]$ of degree 2?

(ii) How many monic irreducible polynomials are there in $\mathbb{F}_p[x]$ of degree 2?

(iii) How many monic polynomials are there in $\mathbb{F}_p[x]$ of degree 3 that have a multiple root?

(iv) How many monic irreducible polynomials are there in $\mathbb{F}_p[x]$ of degree 3?
2. Let $F$ be a field. Prove that in the polynomial ring $F[x]$ there are infinitely many irreducible polynomials.

3. Suppose $\alpha$ is a complex number that is algebraic over $\mathbb{Q}$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd. Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$. 
4. For a prime number \( p \) and a nonzero \( a \in \mathbb{F}_p \), where \( \mathbb{F}_p \) is the field with \( p \) elements, prove that the polynomial \( x^p - x + a \) is irreducible and separable over \( \mathbb{F}_p \).

5. Let \( \omega \in \mathbb{C} \) be a primitive 10-th root of unity.
   (i) What is \([\mathbb{Q}(\omega) : \mathbb{Q}]\)?

   (ii) Diagram the lattice of subfields of \( \mathbb{Q}(\omega) \) giving generators for each.
6. Let $G$ be a finite group and let $p$ be a prime number dividing $|G|$. Let $\mathcal{S}$ denote the set of $p$-tuples of elements of $G$ the product of whose coordinates is 1: thus $\mathcal{S} = \{(x_1, x_2, \ldots, x_p) \mid x_i \in G \text{ and } x_1 x_2 \cdots x_p = 1\}$.

(i) What is the cardinality of $\mathcal{S}$?

(ii) For $\alpha, \beta \in \mathcal{S}$, define $\alpha \sim \beta$ if $\beta$ is a cyclic permutation of $\alpha$. What is needed for $\sim$ to be an equivalence relation on $\mathcal{S}$?

(iii) Assuming $\sim$ is an equivalence relation, which equivalence classes with respect to $\sim$ contain exactly one element?

(iv) What integers are the order of an equivalence class with respect to $\sim$? Justify your answer.

(v) Prove that $G$ has an element of order $p$. 

7. Let $F$ be an infinite field and let $K/F$ be a finite algebraic field extension.

(i) If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of $K$ that contain $F$.

(ii) If there are only finitely many subfields of $K$ that contain $F$, prove that $K = F(\alpha)$ for some $\alpha \in K$. 
8. Let \( \mathbb{Z} \) denote the ring of integers and let \( x \) be an indeterminate over \( \mathbb{Z} \).

(i) Is every ideal of \( \mathbb{Z}[x]/(x^2 - 1) \) principal? Justify your answer.

(ii) Is every ideal of \( \mathbb{Z}[x]/(x^2) \) principal? Justify your answer.

(iii) Is every ideal of \( \mathbb{Z}[x]/(15) \) principal? Justify your answer.
9. Let $\mathbb{Z}$ denote the ring of integers and let $x$ be an indeterminate over $\mathbb{Z}$. Diagram the lattice of ideals of the ring $\mathbb{Z}[x]/(6, x^3)$.

10. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic polynomial of degree 5 that is reducible in $\mathbb{Q}[x]$ and let $L/\mathbb{Q}$ be a splitting field of $f(x)$. List all positive integers $n$ that are possibly equal to $[L : \mathbb{Q}]$. 
Suppose $L/K$ is a separable normal field extension with $[L : K] = 21$.

(i) What integers $n$ are possibly equal to the degree of a monic irreducible polynomial $f(x) \in K[x]$ for which $L/K$ is a splitting field of $f(x)$?

(ii) If the Galois group of $L/K$ is known to be an abelian group, what integers $n$ are possibly equal to the degree of a monic irreducible polynomial $f(x) \in K[x]$ for which $L/K$ is a splitting field of $f(x)$? Justify your answer.
12. Let $G$ be a finite group with $|G| = n$.

(i) Prove that $G$ is isomorphic to a subgroup of the symmetric group $S_n$.

(ii) Is $G$ isomorphic to a subgroup of the alternating group $A_m$ for some positive integer $m$? Justify your answer.
13. Suppose $G_i$ is a group and $H_i$ is a normal subgroup of $G_i$, $i = 1, 2$. Let “$\cong$” denote “is group isomorphic to”.

(i) If $H_1 \cong H_2$ and $G/H_1 \cong G/H_2$, does it follow that $G_1 \cong G_2$? Justify your answer.

(ii) If $G_1 \cong G_2$ and $H_1 \cong H_2$, does it follow that $G_1/H_1 \cong G_2/H_2$? Justify your answer.

(iii) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, does it follow that $H_1 \cong H_2$? Justify your answer.
14. Let $R$ be an integral domain and let $r \in R$ be a nonzero nonunit.

(i) What does it mean for $r$ to be irreducible?

(ii) What does it mean for $r$ to be prime?

(iii) Prove that if $r$ is prime, then $r$ is irreducible.

15. Suppose $H$ is a subgroup of a group $G$ such that $[G : H] = 6$. Prove there exists a normal subgroup $N$ of $G$ such that $N \subseteq H$ and $[G : N] \leq 6!$. 