J. Lipman **Math 553 Qualifying Examination** **August, 1999**

Each problem 1–4 is worth 10 points, and #5 is worth 20. In working any part of a problem you may assume the preceding parts, even if you haven't done them.

Please begin your answer to each question 1–5 on a new sheet.

1. Let G be a finite group and let p be a prime integer dividing the order of G.

(a) Define what is meant by a Sylow p-subgroup of G.

(b) Let S be a Sylow p-subgroup of G and let N be the normalizer of S. Prove that S is a *normal* subgroup of G if and only if $[G : N] \le p$. (You may quote text-book theorems without proof, so your answer should be only a few sentences long.)

2. Let a be a nonzero element in an integral domain R. Suppose that

$$a = p_1 p_2 \dots p_s = q_1 q_2 \dots q_t$$

where the p_i are *prime* elements (i.e., the ideals $p_i R$ are prime), and the q_j are *irreducible* (i.e., q_j is a nonunit such that in any factorization $q_j = uv$, one of u, v, must be a unit). Prove that $s = t$ and that there is a permutation π of $\{1, 2, \dots, s\}$ such that $p_i R = q_{\pi(i)} R$ for all $i = 1, 2, \dots, s$.

3. Let G be a nonabelian group of order 12, and suppose there exists a *surjective* homomorphism $G \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ (where $\mathbb{Z}_2$ is a cyclic group of order 2).

(a) Prove that G is isomorphic to the dihedral group $\mathbf{D}_6$.

(b) Prove that G has exactly three subgroups of order 6, just one of which is cyclic.

4. Let $\mathbb{F}_5$ be the field of cardinality 5.

(a) Prove that if L is a finite field of characteristic 5 and cardinality $|L|$ then L contains a primitive 18-th root of unity ζ if and only if 18 divides $|L| - 1$; and deduce that $[\mathbb{F}_5(\zeta) : \mathbb{F}_5] = 6$.

(b) Show that the polynomial $X^6 - X^3 + 1$ is irreducible over $\mathbb{F}_5$.

(c) Prove that the polynomial $X^6 + 4X^3 + 1$ is irreducible over the field $\mathbb{Q}$ of rational numbers.

5. *In working this problem you may assume, without proof, results in the preceding problems 3 and 4.*

Let $L \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ (field of rational numbers) of the polynomial $f(X) = X^6 + 4X^3 + 1$.

(a) Show that $\sqrt{3} \in L$, and that $[L : \mathbb{Q}(\sqrt{3}] \le 6$.

(b) Show that L contains $\omega := e^{2\pi i/3}$.

(c) Prove that f has a real root x; and show that there is an automorphism α of L such that $\alpha(\omega) = \bar{\omega}$ (the complex conjugate of ω) and $\alpha(x) = \omega/x$.

(d) With γ denoting "complex conjugation" show that $\alpha\gamma \ne \gamma\alpha$.

(e) Prove that the galois group $G(L/\mathbb{Q})$ is isomorphic to the dihedral group $\mathbf{D}_6$.

(f) Prove that the order of α (see (c)) in the galois group $G(L/\mathbb{Q})$ is 6.

(g) Show that the field $F \subset L$ is such that $[F : \mathbb{Q}] = 2$ and $G(L/F)$ is cyclic if and only if $F = \mathbb{Q}(i)$ (where $i^2 = -1$).