QUALIFYING EXAMINATION

AUGUST 2000

MATH 553 - K. Matsuki

Write down answers to the following questions with your reasoning. If your reasoning is correct, even when your final answer happens to be wrong, you will get a substantial amount of credit. On the other hand, providing a final answer without any reasoning will not get full credit.

1. Let $G = S_3$ be the symmetric group of degree 3, i.e., the group of permutations of 3 distinct numbers.
   i) (10 points) What is the total number of subgroups of G (including G itself and the trivial group consisting only of the identity) ?
   ii) (15 points) What is the total number of endomorphisms of G (i.e., group homomorphisms from G to G itself) ?
   iii) (10 points) What is the total number of automorphisms of G (i.e., bijective endomorphisms of G) ?
   iv) (15 points) What is the total number of subgroups of D_{30} (the dihedral group of order 30, which is the group of symmetrics of the regular 15-gon) which are isomorphic to S_3?

2. Let G be a finite group and p a prime integer.
   i) (5 points) Give the definition of H being a Sylow p-subgroup of G.
   ii (15 points) Let H be a Sylow p-subgroup of G. Show that $N \cap H$ is a Sylow p-subgroup of N for any normal subgroup N of G.

3. Let $R = \{a + b \cdot i; a, b \in \mathbb{Z}, i^2 = -1\}$ be the ring of Gaussian integers.
   i) (15 points) Show that R is a Unique Factorization Domain.
   ii) (15 points) Factor the number 70 into prime elements in the ring R. (Verify that each factor in the chosen factorization is a prime element in R.)

4. Let $\zeta = exp(\frac{2\pi\sqrt{-1}}{5})$ be a primitive 5-th root of unity.
   i) (10 points) Find the minimal polynomial of ζ over the field of rational numbers $\mathbb{Q}$.
   ii) (10 points) Determine the Galois group $G(\mathbb{Q}(\zeta)/\mathbb{Q})$ of the extension $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   iii) (20 points) Find all the intermediate fields between $\mathbb{Q}(\zeta)$ and $\mathbb{Q}$ together with their generators over $\mathbb{Q}$.

5. Let $\mathbb{Q}(\sqrt[3]{2}, \omega)$ be an extension of $\mathbb{Q}$, where $\omega = exp(\frac{2\pi\sqrt{-1}}{3})$ is a primitive 3rd root of unity.

   i) (15 points) Determine the Galois group $G(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q})$ of the extension $\mathbb{Q}(\sqrt[3]{2}, \omega)$ over $\mathbb{Q}$.

   ii) (15 points) Find an element $\gamma \in \mathbb{Q}(\sqrt[3]{2}, \omega)$ such that $\mathbb{Q}(\sqrt[3]{2}, \omega) = \mathbb{Q}(\gamma)$. (Give reasoning why your choice of γ satisfies the required property.)

6. Let $f(X) = X^4 + 1 \in \mathbb{Z}[X]$ be a polynomial over $\mathbb{Z}$.

   i) (15 points) Show that $f(X)$ divides $X^{p^2} - X$ for any prime integer $p > 2$.

   ii) (15 points) Show that $f(X)$, considered as a polynomial in $\mathbb{F}_p[X]$ where $\mathbb{F}_p = \mathbb{Z}/(p)$ is the finite field with p elements, is reducible for any prime integer $p > 2$.