QUALIFYING EXAMINATION
January 2000
Math 553 – Prof. Goldberg

Instructions: Give a complete answer to each question. You may use any known result (be clear about what results you are using). When working part of a problem, you may assume the answer to the preceding parts.

1. (12 points) Find all groups of order $7 \cdot 11^3$ which have a cyclic subgroup of order $11^3$.

2. Let $R$ be a ring with identity 1 and consider the following two conditions:
   (I) If $a, b \in R$ and $ab = 0$, then $ba = 0$;
   (II) If $a, b \in R$ and $ab = 1$, then $ba = 1$;
(a) (10 points) Show that I implies II.
(b) (8 points) Show by example that II does not imply I.

3. Let $F$ be a field. Suppose that $E/F$ is a Galois extension, and that $L/F$ is an algebraic extension with $L \cap E = F$. Let $EL$ be the composite field, i.e., the subfield of an algebraic closure $\bar{F}$ of $F$ generated by $E$ and $L$.
   (a) (10 points) Show $EL/L$ is a Galois extension.
   (b) (8 points) Show that there is an injective homomorphism
   \[ \varphi : \text{Gal}(EL/L) \to \text{Gal}(E/F). \]
   Find the fixed field of the image of $\varphi$.
   (c) (6 points) Show that $[EL : L] = [E : F]$.
   (d) (6 points) Give an example to show that the conclusion of (c) is false if we do not assume that $E/F$ is Galois.

4. (12 points) Let $G$ be a finite group. Let $p$ be a prime and suppose that $|G| = p^k m$, with $k \geq 1$ and $p \nmid m$. Let $X$ be the collection of all subsets of $G$ of order $p^k$. Then $G$ acts on $X$ by left multiplication, i.e., $g \cdot A = \{ga | a \in A\}$. For $A \in X$, denote by $H_A$ the stabilizer in $G$ of $A$. Show that $|H_A||p^k$.

5. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the ring consisting of polynomials with rational coefficients whose constant term is an integer.
   (a) (8 points) Prove that $R$ is an integral domain, with units $\pm 1$.
   (b) (8 points) Show that $x$ is not an irreducible element of $R$.
   (c) (12 points) Let $(x) = Rx$ be the ideal of $R$ generated by $x$. Describe $R/(x)$ and show that $R/(x)$ is not an integral domain. What can you conclude about $x$?