Let $\mathbb{Z}$ denote the ring of integers and $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the fields of rational, real and complex numbers, respectively.

(15) 1. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

   (i) Let $\text{Aut}(\overline{\mathbb{Q}})$ denote the group of automorphisms of $\overline{\mathbb{Q}}$. Is the group $\text{Aut}(\overline{\mathbb{Q}})$ finite or infinite? Justify your answer.

   (ii) Is the group $\text{Aut}(\overline{\mathbb{Q}})$ abelian? Justify your answer.

   (iii) Is the group $\text{Aut}(\mathbb{R})$ of automorphisms of $\mathbb{R}$ abelian? Justify your answer.
(18) 1. (continued)

(iv) Prove the existence of a subfield $K$ of $\mathbb{Q}$ that is maximal with respect to $\sqrt{2} \not\in K$.

(v) Prove the existence of subfields $K_1$ and $K_2$ of $\mathbb{Q}$ each maximal with respect to not containing $\sqrt{2}$ such that $[K_1(\sqrt{2}) : K_1] = 3$ and $[K_2(\sqrt{2}) : K_2] = 2$.

(vi) With $K$ as in (iv), if $L/K$ is a finite algebraic field extension, prove that $L/K$ is cyclic and that either $[L : K] = 3^n$ or $[L : K] = 2^n$ for some $n \in \mathbb{N}$. 


Recall that if $R$ and $S$ are rings, then $R \times S = \{(r, s) \mid r \in R, s \in S\}$ is a ring where addition and multiplication in $R \times S$ are defined componentwise.

(6) 2. Describe all the prime ideals of $\mathbb{Z} \times \mathbb{Z}$.

(12) 3. Consider the polynomial ring $\mathbb{Z}[x]$.
   (i) Define $\phi_1 : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi_1(f(x)) = f(1)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_1$.

   (ii) Define $\phi_2 : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi(f(x)) = f(2)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_2$.

   (iii) Define $\phi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$ by $\phi(f(x)) = (f(1), f(2))$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi$.

   (iv) Prove or disprove that $\phi$ is surjective.
(18) 4. Let $K/F$ be a finite separable algebraic field extension and let $\alpha \in K$.
   (i) Define the norm $N_{K/F}(\alpha)$ of $\alpha$ from $K$ to $F$.

   (ii) Prove that $N_{K/F}(\alpha) \in F$.

   (iii) Define the trace $Tr_{K/F}(\alpha)$ of $\alpha$ from $K$ to $F$.

   (iv) Prove that $Tr_{K/F}(\alpha) \in F$.

   (v) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(\sqrt[3]{2})$ and $Tr_{K/\mathbb{Q}}(\sqrt[3]{2})$.

   (vi) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(3)$ and $Tr_{K/\mathbb{Q}}(3)$. 
5. True or false: If \( f(x), g(x) \in \mathbb{Q}[x] \) are irreducible polynomials that have the same splitting field, then \( \text{deg } f = \text{deg } g \). Justify your answer.

6. (i) Give generators for each maximal ideal of the polynomial ring \( \mathbb{Z}[x] \) that contains the ideal \( (15, x^2 - 3) \).

(ii) Diagram the lattice of ideals of the polynomial ring \( \mathbb{Z}[x] \) that contain the ideal \( (15, x^2 - 3) \).
7. Suppose $L/\mathbb{Q}$ is a finite algebraic field extension with $[L : \mathbb{Q}] = 4$. Is it possible that there exist exactly two subfields $K_1$ and $K_2$ of $L$ for which $[L : K_i] = 2$? Justify your answer.

8. Suppose $L/\mathbb{Q}$ is a finite algebraic field extension with $[L : \mathbb{Q}] = 4$. Is it possible that there does not exist a subfield $K$ of $L$ for which $[L : K] = 2$? Justify your answer.
9. Let \( \omega \in \mathbb{C} \) be a primitive 12-th root of unity.

(i) What is \([Q(\omega) : Q]\)?

(ii) List the distinct conjugates of \( \omega + \omega^{-1} \) over \( Q \).

(iii) What is the group \( \text{Aut}(Q(\omega + \omega^{-1})/Q) \)? Is \( Q(\omega + \omega^{-1}) \) Galois over \( Q \)?

(iv) Diagram the lattice of subfields of \( Q(\omega) \) giving generators for each.
10. Let $F$ be a field. For each nonconstant monic polynomial $f = f(x) \in F[x]$, let $x_f$ be an indeterminate. Consider the polynomial ring $R = F[\{x_f\}]$, and let $I$ be the ideal of $R$ generated by the polynomials $f(x_f)$, where $f$ varies over all the nonconstant monic polynomials in $F[x]$.

(i) Prove that $I \neq R$.

(ii) Prove that there exists an extension field $K$ of $F$ in which each nonconstant monic polynomial $f \in F[x]$ has a root.

11. Is $\mathbb{Q}(\sqrt{2})$ the fixed field of an automorphism of $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$? Justify your answer.
12. Let $K / \mathbb{Q}$ be the splitting field of the polynomial $x^9 - 1 \in \mathbb{Q}[x]$.

(i) Diagram the lattice of subfields of $K / \mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

(ii) Let $\alpha \in \mathbb{R}$ be a root of the polynomial $x^9 - 2 \in \mathbb{Q}[x]$ and let $L$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. What is $[L : \mathbb{Q}(\alpha)]$?

(iii) Diagram the lattice of subfields $F$ of $L$ that contain $\mathbb{Q}(\alpha)$. For each such field $F$, list $[F : \mathbb{Q}(\alpha)]$ and give a generator for $F$ over $\mathbb{Q}(\alpha)$. 
(7) 13. Suppose \( f(x) \in \mathbb{Q}[x] \) is a monic polynomial of degree \( n \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) are the roots of \( f(x) \). Let \( G \) be the Galois group of \( f(x) \) over \( \mathbb{Q} \). Prove that \( f(x) \) is irreducible in \( \mathbb{Q}[x] \) if and only if the action of \( G \) on \( \{\alpha_1, \ldots, \alpha_n\} \) is transitive.

(7) 14. Let \( G \) act as a transitive permutation group on the finite set \( A \) with \( |A| > 1 \). Prove that there exists \( g \in G \) such that \( g(a) \neq a \) for all \( a \in A \).
15. Let $R$ be an integral domain. 

(i) Define $R$ is a \textit{unique factorization domain (UFD)}.

(ii) Give an example of an integral domain that is not a UFD.

(iii) Assume that $R$ is a subring of a UFD $S$ and that every unit of $S$ is contained in $R$. Also assume for all $a$ and $b$ in $S$, if $ab \in R$, then $a$ and $b$ are in $R$. Prove that $R$ is a UFD.

(iv) If the polynomial ring $R[x]$ is a UFD, prove that $R$ is a UFD.
16. Let $R$ be a commutative ring with identity $1 \neq 0$ and let $f(x)$ and $g(x)$ be polynomials in $R[x]$. Let $c(f), c(g)$ denote the ideals of $R$ generated by the coefficients of $f(x), g(x)$, respectively. If $c(f) = c(g) = R$, prove that the ideal $c(fg)$ generated by the coefficients of the product $f(x)g(x)$ is also equal to $R$.

17. Assume that $F$ is a field of characteristic zero and that $K/F$ is an algebraic field extension. If each nonconstant polynomial in $F[x]$ has at least one root in $K$, prove that $K$ is algebraically closed.