QUALIFYING EXAMINATION
AUGUST 2006
MA 553

1. (15 points) Let G be a group of order $2n$, where n is odd. Show that G has a subgroup of index 2. (Hint: embed G into S_{2n}.)

2. (14 points) Let G be a group of odd order and let H be a normal subgroup of order 5. Show that H is in the center of G.

3. (14 points) Show that up to isomorphism, there are at most two groups of order 147 having an element of order 49.

4. (14 points) Let R be a principal ideal domain and m a maximal ideal of the polynomial ring $R[X]$ with $m \cap R \neq 0$. Show that $m = (p, f)$ for some prime element p of R and some monic irreducible polynomial f in $R[X]$.

5. (14 points) Let $k \subset K$ be a normal extension of fields of characteristic $p > 0$ with $G = \mathrm{Aut}_k(K)$. Show that the extension $k \subset K^G$ is purely inseparable.

6. (15 points) Let $k \subset K_1$ and $k \subset K_2$ be finite Galois extensions contained in a common field, and write $K = K_1K_2$.

(a) Show that the extension $k \subset K$ is finite Galois.

(b) Show that the Galois group $G(K/k)$ is isomorphic to the subgroup $H = \{(\sigma, \tau) | \sigma_{|K_1 \cap K_2} = \tau_{|K_1 \cap K_2}\}$ of $G(K_1/k) \times G(K_2/k)$.

7. (14 points) Let p be a prime number, $\zeta \in \mathbb{C}$ a primitive p^{th} root of unity and $K = \mathbb{Q}(\zeta)$. Determine those p for which K has a unique maximal proper subfield $k \subsetneq K$.