QUALIFYING EXAMINATION
AUGUST 2006
MA 553

1. (15 points) Let $G$ be a group of order $2n$, where $n$ is odd. Show that $G$ has a subgroup of index 2. (Hint: embed $G$ into $S_{2n}$.)

2. (14 points) Let $G$ be a group of odd order and let $H$ be a normal subgroup of order 5. Show that $H$ is in the center of $G$.

3. (14 points) Show that up to isomorphism, there are at most two groups of order 147 having an element of order 49.

4. (14 points) Let $R$ be a principal ideal domain and $m$ a maximal ideal of the polynomial ring $R[X]$ with $m \cap R \neq 0$. Show that $m = (p, f)$ for some prime element $p$ of $R$ and some monic irreducible polynomial $f$ in $R[X]$.

5. (14 points) Let $k \subset K$ be a normal extension of fields of characteristic $p > 0$ with $G = \text{Aut}_k(K)$. Show that the extension $k \subset K^G$ is purely inseparable.

6. (15 points) Let $k \subset K_1$ and $k \subset K_2$ be finite Galois extensions contained in a common field, and write $K = K_1K_2$.

   (a) Show that the extension $k \subset K$ is finite Galois.

   (b) Show that the Galois group $G(K/k)$ is isomorphic to the subgroup $H = \{ (\sigma, \tau) \mid \sigma|_{K_1 \cap K_2} = \tau|_{K_1 \cap K_2} \} \times G(K_2/k)$.

7. (14 points) Let $p$ be a prime number, $\zeta \in \mathbb{C}$ a primitive $p^{th}$ root of unity and $K = \mathbb{Q}(\zeta)$. Determine those $p$ for which $K$ has a unique maximal proper subfield $k \subsetneq K$. 

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