This exam is to be done in two hours in one continuous sitting. Begin each question on a new sheet of paper. In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work: give definitions of all terms you state, provide references for all theorems you quote, and prove all statements you claim.

Problem 1. Let \((G, \circ)\) be a group, \((H, \ast)\) be an abelian group, and \(\varphi : G \rightarrow H\) be a group homomorphism. If \(N\) is a subgroup such that \(\ker(\varphi) \leq N \leq G\), show that \(N \trianglelefteq G\) is a normal subgroup. [10 points]

Problem 2. Let \((G, \circ)\) be a finite abelian group of even order \(|G| = 2k\).
   a. For \(k\) odd, show that \(G\) has exactly one element of order 2. [5 points]
   b. Does the same happen for \(k\) even? Prove or give a counterexample. [5 points]

Problem 3. Let \((G, \circ)\) be a finite group of odd order, and \(H \trianglelefteq G\) be a normal subgroup of prime order \(|H| = 17\). Show that \(H \leq \mathbb{Z}(G)\). [10 points]

Problem 4. Let \((R, +, \cdot)\) and \((S, +, \cdot)\) be commutative rings with \(1 \neq 0\), and let \(\varphi : R \rightarrow S\) be a surjective ring homomorphism. Assuming that \(R\) is local, i.e., it has a unique maximal ideal, show that \(S\) is also local. [10 points]

Problem 5. Let \((R, +, \cdot)\) be a Principal Ideal Domain.
   a. Show that every maximal ideal in \(R\) is a prime ideal. [5 points]
   b. Must every prime ideal in \(R\) be a maximal ideal? Prove or give a counterexample. [5 points]

Problem 6. Let \((L/F)\) be a Galois extension of degree \([L:F] = 2p\), where \(p\) is an odd prime.
   a. Show that \(L/F\) is a Galois extension of degree \([L:F] = 2p\), where \(p\) is an odd prime.
   b. Show that \(L/F\) is a Galois extension of degree \([L:F] = 2p\), where \(p\) is an odd prime.

Problem 7. Fix a prime \(p\), and consider the Artin-Schreier polynomial \(f(x) = x^p - x - 1\).
   a. Let \(\mathbb{F}_p(f)\) be the splitting field of \(f(x)\) over \(\mathbb{F}_p\). Show that \(\text{Gal}(\mathbb{F}_p(f)/\mathbb{F}_p) \cong \mathbb{Z}_p\). [5 points]
   b. Prove that \(f(x)\) is irreducible in \(\mathbb{Z}[x]\). [5 points]

Problem 8. Determine the Galois group of the splitting field over \(\mathbb{Q}\) of \(f(x) = x^4 + 4\). [10 points]