1. (20 pts) Let G be a nontrivial finite group.

(a) What is meant by a composition series for G?

(b) State the Jordan-Hölder theorem.

(c) What does it mean for G to be simple?

(d) What does it mean for G to be solvable?

(e) Give an example of a simple group that is not solvable.

2. (16 pts)

(a) Does the symmetric group S_5 have a subgroup of order 20? Justify your answer.

(b) Does the symmetric group S_5 have a subgroup of order 15? Justify your answer.

(c) Let G be a finite group. Is G isomorphic to a subgroup of the alternating group A_n for some positive integer n? Justify your answer.

(d) Determine the number of elements of order 2 in the alternating group A_5.

3. (8 pts) Suppose σ is an element of order 2 in the alternating group A_n. Prove or disprove that there exists $\tau \in S_n$ such that $\tau^2 = \sigma$.

4. (8 pts) Find all finite groups that have exactly three conjugacy classes.

5. (12 pts) Let G be a finite group of order pqr, where $p > q > r$ are prime.

(a) If G fails to have a normal subgroup of order p, determine the number of elements in G of order p.

(b) If G fails to have a normal subgroup of order q, prove that G has at least q^2 element of order q.

(c) Prove that G has a nontrivial normal subgroup.

6. (6 pts) Give an example of a commutative ring R with identity $1 \neq 0$ that has ideals I and J such that $\{ab \mid a \in I, b \in J\}$ is not an ideal of R. Justify your answer.

7. (12 pts) Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(35,\ x^2 - 2)$. Give generators for each such ideal

8. (8 pts) Prove or disprove that a nonzero prime ideal P of a principal ideal domain R is a maximal ideal.

9. (7 pts) Prove that the polynomial

$$f_n(x) = (x-1)(x-2)\cdots(x-n) - 1$$

is irreducible over $\mathbb{Z}$ for each integer $n \geq 1$.

10. (7 pts) Prove that the polynomial

$$g_n(x) = (x-1)(x-2)\cdots(x-n) + 1$$

is irreducible over $\mathbb{Z}$ for each positive integer $n \neq 4$.

11. (5 pts) State Gauss' Lemma.

12. (8 pts) Assume that $f(x)$ and $g(x)$ are polynomials in $\mathbb{Q}[x]$ and that $f(x)g(x) \in \mathbb{Z}[x]$. Prove that the product of any coefficient of $f(x)$ with any coefficient of $g(x)$ is an integer.

13. (5 pts) True or false: If $f(x), g(x) \in \mathbb{Q}[x]$ are irreducible polynomials that have the same splitting field, then $\deg f = \deg g$. Justify your answer.

14. (15) Let p be a prime integer and let $\mathbb{F}_p$ denote the field with p elements.

(a) Prove or disprove that every finite algebraic extension field of $\mathbb{F}_p$ is Galois.

(b) Let K and L be finite algebraic field extensions of $\mathbb{F}_p$. If $[K : \mathbb{F}_p] \leq [L : \mathbb{F}_p]$, does it follow that K is isomorphic to a subfield of L? Justify your answer.

(c) Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. If E is a subfield of $\overline{\mathbb{F}_p}$ and $[E : \mathbb{F}_p] = \infty$, does it follow that $E = \overline{\mathbb{F}_p}$? Justify your answer.

15. (8 pts) Let F be a field and let K_1/F and K_2/F be finite Galois extensions contained in an algebraic closure $\overline{F}$ of F. Prove or disprove that the composite field K_1K_2 is Galois over F.

16. (8 pts) Let $L/\mathbb{Q}$ be the Galois closure of the simple algebraic field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$. Let p be a prime that divides $[L : Q]$. Prove that there exists a subfield F of L such that $[L : F] = p$ and $L = F(\alpha)$.

17. (10 pts) Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{R}$.

(a) What is the minimal polynomial for α over $\mathbb{Q}$?

(b) List the conjugates of α over $\mathbb{Q}$.

(c) List the conjugates of α over $\mathbb{Q}(\sqrt{2})$.

(d) Is $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois ? Justify your answer.

18. (10 pts) Let $\beta = \sqrt{1+\sqrt{3}} \in \mathbb{R}$.

(a) What is the minimal polynomial for β over $\mathbb{Q}$?

(b) List the conjugates of β over $\mathbb{Q}$.

(c) List the conjugates of β over $\mathbb{Q}(\sqrt{3})$.

(d) Is $\mathbb{Q}(\beta)/\mathbb{Q}$ Galois ? Justify your answer.

19. (8 pts) Let F be a subfield of the field $\mathbb{C}$ of complex numbers and let $K \subseteq \mathbb{C}$ be an algebraic field extension of F having the property that each nonconstant polynomial in $F[x]$ has at least one root in K. Prove that K is algebraically closed.

20. (6 pts) Give an example of a finite algebraic field extension L/K for which there exist infinitely many intermediate fields between K and L.

21. (8 pts) Let n be a positive integer and d a positive integer that divides n. Suppose $\alpha \in \mathbb{R}$ is a root of the polynomial $x^n - 2 \in \mathbb{Q}[x]$. Prove that there is precisely one subfield F of $\mathbb{Q}(\alpha)$ with $[F : \mathbb{Q}] = d$.

22. (5) Suppose $L/\mathbb{Q}$ is a finite field extension with $[L : \mathbb{Q}] = 4$. Is it possible that there exist precisely two subfields K_1 and K_2 of L for which $[L : K_i] = 2$? Justify your answer.