QUALIFYING EXAMINATION

AUGUST 2009

MA 553

1. (13 points) Let G be a group such that $G/Z(G)$ is Abelian, and let $H \neq \{e\}$ be a normal subgroup of G. Show that $H \cap Z(G) \neq \{e\}$. (Hint: Consider the commutator subgroup G' of G.)

2. (15 points) Let G be a group of order 150. Show that G has a normal subgroup of order 25. (Hint: You may want to show that G has a normal subgroup of order 5 or 25.)

3. (14 points) Show that up to isomorphism, there are at most three non-Abelian groups of order 70.

4. (14 points) Let R be a unique factorization domain with quotient field K, let $K \subset L$ be a field extension, and let α be an element of L that is algebraic over K. Consider the subring $R[\alpha]$ of L. Find an ideal I of the polynomial ring $R[X]$ so that $R[\alpha] \cong R[X]/I$. (Hint: Consider the minimal polynomial of α over K.)

5. (15 points) Let k be a field of characteristic $p > 0$, and let $k \subset K$ be an algebraic field extension of finite inseparable degree.

(a) Show that there exists $e \in \mathbb{N}$ such that $kK^{p^n} = kK^{p^e}$ for every $n \geq e$.

(b) Show that the inseparable degree of $k \subset K$ is $[K : kK^{p^e}]$ for e as in (a).

6. (15 points) Let k be a field, let $f(X) \in k[X]$ be a separable polynomial of degree n whose Galois group is isomorphic to S_n, and let α be a root of $f(X)$ in some algebraic closure $\overline{k}$.

(a) Show that $f(X)$ is irreducible.

(b) Show that $\mathrm{Aut}_k(k(\alpha)) = \{\mathrm{id}\}$ if $n \geq 3$.

(c) Show that $\alpha^n \notin k$ if $n \geq 4$.

7. (14 points) Determine the Galois group (up to isomorphism) of the polynomial $f = X^4 - 4X^2 + 2$ over $\mathbb{Q}$. Find all intermediate fields between $\mathbb{Q}$ and the splitting field of f over $\mathbb{Q}$.